TORIC RINGS, INSEPARABILITY AND RIGIDITY

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Abstract. Let K be a field, H an affine semigroup and $R = K[H]$ its toric ring over K. In this paper we give an explicit description of the $\mathbb{Z}H$-graded components of the cotangent module $T^1(R)$ which classifies the infinitesimal deformations of R. In particular, we are interested in unobstructed deformations which preserve the toric structure. Such deformations we call separations. Toric rings which do not admit any separation are called inseparable. We apply the theory to the edge ring of a finite graph. The coordinate ring of a convex polyomino may be viewed as the edge ring of a special class of bipartite graphs. It is shown that the coordinate ring of any convex polyomino is inseparable. We call a bipartite graph G semi-rigid if $T^1(R)_a = 0$ for all a with $-a \in H$. Here R is the edge ring of G. A combinatorial description of semi-rigid graphs is given. The results are applied to show that for $n = 3$, G_n is semi-rigid but not rigid, while G_n is rigid for all $n \geq 4$. Here G_n is the complete bipartite graph $K_{n,n}$ with one edge removed.

Introduction

In this paper we study infinitesimal deformations and unobstructed deformations of toric rings which preserve the toric structure, and apply this theory to edge ideals of bipartite graphs. Already in [1] and [2], infinitesimal and homogeneous deformations of toric varieties have been considered from a geometric point of view. The view point of this paper is more algebraic and does not exclude non-normal toric rings, having in mind toric rings which naturally appear in combinatorial contexts. This aspect of deformation theory has also been pursued in the papers [4],[5] and [3] where deformations of Stanley-Reisner rings attached to simplicial complexes were studied.

Let K be a field. The infinitesimal deformations of a finitely generated K-algebra R are parameterized by the elements of the cotangent module $T^1(R)$ which in the case that R is a domain is isomorphic to $\mathrm{Ext}^1_R(\Omega_{R/K}, R)$, where $\Omega_{R/K}$ denotes the module of differentials of R over K. The ring R is called *rigid* if $T^1(R) = 0$. We refer the reader to [12] for the theory of deformation.

Let H be an affine semigroup and $K[H]$ its affine semigroup ring. We are interested in the module $T^1(K[H])$. This module is naturally $\mathbb{Z}H$-graded. Here $\mathbb{Z}H$ denotes the associated group of H which for an affine semigroup is a free group of

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finite rank. For each $a \in \mathbb{Z}H$, the a-graded component $T^1(K[H])_a$ of $T^1(K[H])$ is a finite dimensional K-vector space.

In Section 1 we describe the vector space $T^1(K[H])_a$ and provide a method how to compute its dimension. Let $H \subset \mathbb{Z}^m$ with generators $h_1, \ldots, h_n$. Then the associated group $\mathbb{Z}H$ of H is a subgroup of $\mathbb{Z}^m$, and $K[H]$ is the K-subalgebra of the ring $K[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ of Laurent polynomials generated by the monomials $t^{h_1}, \ldots, t^{h_n}$. Here $t^a = t_1^{a(1)} \cdots t_m^{a(m)}$ for $a = (a(1), \ldots, a(m)) \in \mathbb{Z}^m$. Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over K in the indeterminates $x_1, \ldots, x_n$. Then S may be viewed as a $\mathbb{Z}H$-graded ring with $\deg x_i = h_i$, and the K-algebra homomorphism $S \to K[H]$ with $x_i \mapsto t^{h_i}$ is a homomorphism of $\mathbb{Z}H$-graded K-algebras. We denote by I_H the kernel of this homomorphism. The ideal I_H is called the toric ideal associated with H. It is generated by homogeneous binomials. To describe these binomials, consider the group homomorphism $\mathbb{Z}^n \to \mathbb{Z}^m$ with $\varepsilon_i \mapsto h_i$, where $\varepsilon_1, \ldots, \varepsilon_n$ is the canonical basis of $\mathbb{Z}^n$. The kernel L of this group homomorphism is a lattice of $\mathbb{Z}^n$ and is called the *relation lattice* of H. For $v = (v(1), \ldots, v(n)) \in \mathbb{Z}^n$ we define the binomial $f_v = f_{v^+} - f_{v^-}$ with $f_{v^+} = \prod_{i,\, v(i) \geq 0} x_i^{v(i)}$ and $f_{v^-} = \prod_{i,\, v(i) \leq 0} x_i^{-v(i)}$, and let I_L be the ideal generated by the binomials f_v with $v \in L$. It is well known that $I_H = I_L$. Each $f_v \in I_H$ is homogeneous of degree $h(v) = \sum_{i,\, v(i) \geq 0} v(i) h_i$. Let $f_{v_1}, \ldots, f_{v_s}$ be a system of generators of I_H. We consider the $(s \times n)$-matrix

$$A_H = \begin{pmatrix} v_1(1) & v_1(2) & \ldots & v_1(n) \\ v_2(1) & v_2(2) & \ldots & v_2(n) \\ \vdots & \vdots & & \vdots \\ v_s(1) & v_s(2) & \ldots & v_s(n) \end{pmatrix}.$$

Summarizing the results of Section 1, $\dim_K T^1(K[H])_a$ can be computed as follows: let $l = \operatorname{rank} A_H$, l_a be the rank of the submatrix of A_H whose rows are the ith rows of A_H for which $a + h(v_i) \notin H$, and let d_a be the rank of the submatrix of A_H whose columns are the jth columns of A_H for which $a + h_j \in H$. Then

$$\dim_K T^1(K[H])_a = l - l_a - d_a.$$

In Section 2 we introduce the concept of separation for a torsionfree lattice $L \subset \mathbb{Z}^n$. Note that a lattice $L \subset \mathbb{Z}^n$ is torsionfree if and only if it is the relation lattice of some affine semigroup. Given an integer $i \in [n] = \{1, 2, \ldots, n\}$, we say that L admits an *i-separation* if there exists a torsionfree lattice $L' \subset \mathbb{Z}^{n+1}$ of the same rank as L such that $\pi_i(I_{L'}) = I_L$, where $\pi_i \colon S[x_{n+1}] \to S$ is the K-algebra homomorphism which identifies x_{n+1} with x_i. An additional condition makes sure that this deformation which induces an element in $T^1(K(H))_{-h_i}$ is non-trivial, see 2.1 for the precise definition. We say that L is *inseparable*, if for all i, the lattice L admits no i-separation, and we call H and its toric ring *inseparable* if its relation lattice is inseparable. In particular, if the generators of H belong to a hyperplane of $\mathbb{Z}^m$, so that $K[H]$ also admits a natural standard grading, then H is inseparable if $T^1(K[H])_{-1} = 0$. In general, the converse is not true since the infinitesimal deformation given by non-zero element of $T^1(K[H])_{-1}$ may be obstructed. We demonstrate this theory and

show that a numerical semigroup generated by three elements which is not a complete intersection is i-separable for $i = 1, 2, 3$, while if it is a complete intersection it is i-separable for at least two $i \in \{1, 2, 3\}$. For the proof of this fact we use the structure theorem of such semigroups given in [8].

The last Section 3 is devoted to the study of $T^1(R)$ when R is the edge ring of a bipartite graph. This class of rings has been well studied in combinatorial commutative algebra, see e.g. [11] and [13]. For a given simple graph G of the vertex set $[n]$ one considers the edge ring $R = K[G]$ which is the toric ring generated over K by the monomials $t_i t_j$ for which $\{i, j\}$ is an edge of G. Viewing the edge ring a semigroup ring $K[H]$, the edges e_i of G correspond the generators h_i of the semigroup H. We say that G is inseparable if the corresponding semigroup is *inseparable*. The main result of the first part of this section is a combinatorial criterion for G being inseparable. Let C be a cycle of G and e a chord of G. Then e splits C into two disjoint connected components C_1 and C_2 which are obtained by restricting C to the complement of e. A path P of G is called a *crossing path* of C with respect to e if one end of P belongs to C_1 and the other end to C_2. Now the criterion (Corollary 3.5) says that a bipartite graph G is inseparable if and only if for any cycle C which has a unique chord e, there exists a crossing path of C with respect to e. In particular, if no cycle has a chord, then G is inseparable. By using this criterion we show in Theorem 3.6 that the coordinate ring of any convex polyomino, which may be interpreted as a special class of edge rings, is inseparable.

For the rest of this section we consider the semi-rigidity and rigidity of bipartite graphs. We call H *semi-rigid* if $T^1(K(H))_{-a} = 0$ for all $a \in H$, and characterize in Theorem 3.10 semi-rigidity of bipartite graphs in terms of the non-existence of certain constellations of edges and cycles of the graph. To classify rigidity of bipartite graphs is much more complicated, and we do not have a general combinatorial criterion for when a bipartite graph is rigid. Instead we consider for each $n \geq 3$ the graph G_n which is obtained by removing an edge from the complete bipartite graph $K_{n,n}$. It is shown in Proposition 3.12 that for $n = 3$, G_n is not rigid while for $n \geq 4$, G_n is rigid. It remains a challenging problem to classify all rigid bipartite graphs.

1. T^1 for toric rings

Let H be an *affine semigroup*, that is, a finitely generated subsemigroup of $\mathbb{Z}^m$ for some $m > 0$. Let $h_1, \ldots, h_n$ be the minimal generators of H, and fix a field K. The toric ring $K[H]$ associated with H is the K-subalgebra of the ring $K[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ of Laurent polynomials generated by the monomials $t^{h_1}, \ldots, t^{h_n}$. Here $t^a = t_1^{a(1)} \cdots t_m^{a(m)}$ for $a = (a(1), \ldots, a(m)) \in \mathbb{Z}^m$.

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over K in the variables $x_1, \ldots, x_n$. The K-algebra $R = K[H]$ has a presentation $S \to R$ with $x_i \mapsto t^{h_i}$ for $i = 1, \ldots, n$. The kernel $I_H \subset S$ of this map is called the *toric ideal* attached to H. Corresponding to this presentation of $K[H]$ there is a presentation $\mathbb{N}^n \to H$ of H which can be extended to the group homomorphism $\mathbb{Z}^n \to \mathbb{Z}^m$ with $\varepsilon_i \mapsto h_i$ for $i = 1, \ldots, n$, where $\varepsilon_1, \ldots, \varepsilon_n$ denotes the canonical basis of $\mathbb{Z}^n$. Let $L \subset \mathbb{Z}^n$ be the kernel of this

group homomorphism. The lattice L is called the *relation lattice* of H. Note that L is a free abelian group and $\mathbb{Z}^n/L$ is torsion-free.

For a vector $v \in \mathbb{Z}^n$ with $v = (v(1), \dots, v(n))$, we set

$$v_+ = \sum_{i,\, v(i) \geq 0} v(i)\varepsilon_i \quad \text{and} \quad v_- = \sum_{i,\, v(i) \leq 0} -v(i)\varepsilon_i.$$

Then $v = v_+ - v_-$. It is a basic fact and well-known (see e.g. [7]) that I_H is generated by the binomials f_v with $v \in L$, where $f_v = x^{v_+} - x^{v_-}$.

We define an H-grading on S by setting $\deg x_i = h_i$. Then I_H is a graded ideal with $\deg f_v = h(v)$, where

$$h(v) = \sum_{i,\, v(i) \geq 0} v(i)h_i \; (= \sum_{i,\, v(i) \leq 0} -v(i)h_i). \tag{1}$$

Let $v_1, \dots, v_r$ be a basis of L. Since I_H is a prime ideal we may localize S with respect to this prime ideal and obtain

$$I_H S_{I_H} = (f_{v_1}, \dots, f_{v_r})S_{I_H}.$$

In particular, we see that

$$\text{height}\, I_H = \text{rank}\, L. \tag{2}$$

Let $\Omega_{R/K}$ be the module of differentials of R over K. Since R is a domain, the cotangent module $T^1(R)$ is isomorphic to $\text{Ext}^1_R(\Omega_{R/K}, R)$, and since R is H-graded it follows that $\Omega_{R/K}$ is H-graded as well, and hence $\text{Ext}^1_R(\Omega_{R/K}, R)$ and $T^1(R)$ are $\mathbb{Z}H$-graded. Here $\mathbb{Z}H$ denotes the associated group of H, that is, the smallest subgroup of $\mathbb{Z}^m$ containing H. It is our goal to compute the graded components $T^1(R)_a$ of $T^1(R)$ for $a \in \mathbb{Z}H$.

The module of differentials has a presentation

$$\Omega_{R/K} = (\bigoplus_{i=1}^{n} R dx_i)/U,$$

where U is the submodule of the free R-module $\bigoplus_{i=1}^n R dx_i$ generated by the elements df_v with $v \in L$, where

$$df_v = \sum_{i=1}^{n} (\partial f_v / \partial x_i) dx_i.$$

Here $\partial f_v/\partial x_i$ stands for partial derivative of f_v with respect to x_i, evaluated modulo I_H.

One verifies at once that

$$df_v = \sum_{i=1}^{n} v(i) t^{h(v)-h_i} dx_i. \tag{3}$$

For $i \in [n]$, the basis element dx_i of $\Omega_{S/K} \otimes_S R = \bigoplus_{i=1}^n R dx_i$ is given the degree h_i. Then U is an H-graded submodule of $\Omega_{S/K} \otimes_S R$, and $\deg df_v = \deg f_v = h(v)$.

For any $\mathbb{Z}H$-graded R-module M we denote by M^* the graded R-dual $\operatorname{Hom}_R(M,R)$. Then the exact sequence of H-graded R-modules

$$0 \to U \to \Omega_{S/K} \otimes_S R \to \Omega_{R/K} \to 0$$

gives rise to the exact sequence

$$(\Omega_{S/K} \otimes_S R)^* \to U^* \to T^1(R) \to 0$$

of $\mathbb{Z}H$-graded modules. This exact sequence may serve as the definition of $T^1(R)$, namely, to be the cokernel of $(\Omega_{S/K} \otimes_S R)^* \to U^*$.

Let $f_{v_1}, \ldots, f_{v_s}$ be a system of generators of I_H, where we may assume that for $r \le s$, the elements $v_1, \ldots, v_r$ form a basis of L. In general s is much larger than r. Observe that the elements $df_{v_1}, \ldots, df_{v_s}$ form a system of generators of U.

We let F be a free graded R-module with basis $g_1, \ldots, g_s$ such that $\deg g_i = \deg df_{v_i}$ for $i = 1, \ldots, s$, and define the R-module epimorphism $F \to U$ by $g_i \mapsto df_{v_i}$ for $i = 1, \ldots, s$. The kernel of $F \to U$ we denote by C. The composition $F \to \Omega_{S/K} \otimes_S R$ of the epimorphism $F \to U$ with the inclusion map $U \to \Omega_{S/K} \otimes_S R$ will be denoted by δ. We identify $U^* \subset F^*$ with its image in F^*. Then $T^1(R) = U^* / \operatorname{Im} \delta^*$ and U^* is the submodule of F^* consisting of all $\varphi \in F^*$ with $\varphi(C) = 0$.

We first describe the $\mathbb{Z}H$-graded components of U^*. Let $a \in \mathbb{Z}H$. We denote by KL the K-subspace of K^n spanned by $v_1, \ldots, v_s$ and by KL_a the K-subspace of KL spanned by the vectors v_i with $i \notin \mathcal{F}_a$. Here the set $\mathcal{F}_a$ is defined to be

$$\mathcal{F}_a = \{ i \in [s] \colon a + h(v_i) \in H \}.$$

Then we have

Theorem 1.1. *For all $a \in \mathbb{Z}H$, we have*

$$\dim_K (U^*)_a = \dim_K KL - \dim_K KL_a.$$

Proof. Let $\sigma_1, \ldots, \sigma_s$ be the canonical basis of K^s and $W \subset K^s$ be the kernel of the K-linear map $K^s \to KL$ with $\sigma_i \mapsto v_i$ for $i = 1, \ldots, s$.

We will show that

$$(U^*)_a \cong \{ \mu \in K^s \colon\ \mu(i) = 0 \text{ for } i \in [s] \setminus \mathcal{F}_a \text{ and } \langle \mu, \lambda \rangle = 0 \text{ for all } \lambda \in W \}, \tag{4}$$

as K-vector space.

Assuming this isomorphism has been proved, let X_a be the image of $W \subset K^s$ under the canonical projection $K^s \to V_a = \bigoplus_{i \in \mathcal{F}_a} K\sigma_i$. Then (4) implies that $(U^*)_a$ is isomorphic to the orthogonal complement of X_a in V_a. Thus,

$$\dim_K (U^*)_a = |\mathcal{F}_a| - \dim_K X_a. \tag{5}$$

Let $Z_a = \bigoplus_{i \notin \mathcal{F}_a} K\sigma_i$ and Y_a the cokernel of $X_a \to V_a$. Then we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W \cap Z_a & \longrightarrow & Z_a & \longrightarrow & KL_a & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W & \longrightarrow & K^s & \longrightarrow & KL & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_a & \longrightarrow & V_a & \longrightarrow & Y_a & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Now (5) implies that $\dim_K (U^*)_a = \dim_K Y_a$, and the diagram shows that $\dim_K Y_a = \dim_K KL - \dim_K KL_a$.

It remains to prove the isomorphism (4). Observe that $(U^*)_a = \{\varphi \in (F^*)_a \colon \varphi(C) = 0\}$, where C is the kernel of $F \to U$. Let $\varphi \in (F^*)_a$. Then $\varphi = \sum_{i=1}^{s} \varphi(g_i) g_i^*$, where $g_1^*, \dots, g_s^*$ is the basis of F^* dual to $g_1, \dots, g_s$.

Since $\deg g_i^* = -\deg df_{v_i} = -h(v_i)$, it follows that $\varphi \in (F^*)_a$ if and only if $\varphi(g_i) = \mu(i) t^{a+h(v_i)}$ with $\mu(i) \in K$ and $\mu(i) = 0$ if $a + h(v_i) \notin H$. Hence

$$(U^*)_a \cong \{\mu \in K^s \colon \mu(i) = 0 \text{ for } i \in [s] \setminus \mathcal{F}_a \text{ and } (\sum_{i \in \mathcal{F}_a} \mu(i) t^{a+h(v_i)} g_i^*)(C) = 0\}.$$

In order to complete the proof of (4) we only need to prove the following statement:

$$(6) \qquad (\sum_{i=1}^{s} \mu(i) t^{a+h(v_i)} g_i^*)(C) = 0 \text{ if and only if } \langle \mu, \lambda \rangle = 0 \text{ for all } \ \lambda \in W.$$

Let $z \in C_b$ for some $b \in H$. Then $z = \sum_{i \in [s]} \lambda(i) t^{b-h(v_i)} g_i$ with $\lambda(i) \in K$ for $i = 1, \dots, s$ and $\lambda(i) = 0$ if $b - h(v_i) \notin H$ since $z \in \mathcal{F}_a$. Moreover, since $z \in \operatorname{Ker}(F \to U)$ it follows that $\lambda(1) t^{b-h(v_1)} df_{v_1} + \cdots + \lambda(s) t^{b-h(v_s)} df_{v_s} = 0$. This implies that

$$\sum_{\substack{i \in [s] \\ b-h(v_i) \in H}} \sum_{j \in [n]} \lambda(i) t^{b-h(v_i)} v_i(j) t^{h(v_i)-h_j} dx_j = \sum_{j \in [n]} (\sum_{\substack{i \in [s] \\ b-h(v_i) \in H}} \lambda(i) v_i(j) t^{b-h_j}) dx_j = 0.$$

Note that if $b - h_j \notin H$, then for all $i \in [s]$ with $b - h(v_i) \in H$, one has $h(v_i) - h_j \notin H$ and so $v_i(j) = 0$. Here we use the definition of $h(v_i)$, see (1). Therefore, $\sum_{i \in [s], b-h(v_i) \in H} \lambda(i) v_i(j) = 0$ for $j = 1, \dots, n$. This implies $\sum_{i \in [s], b-h(v_i) \in H} \lambda(i) v_i = 0$.

In conclusion we see that

$$\sum_{i \in [s], b-h(v_i) \in H} \lambda(i) t^{b-h(v_i)} g_i \in C_b \quad \text{ if and only if } \quad \sum_{i \in [s], b-h(v_i) \in H} \lambda(i) v_i = 0.$$

This particularly implies that if $z = \sum_{i\in[s]} \lambda(i)t^{b-h(v_i)}g_i \in C_b$, then $\lambda = (\lambda(1), \ldots, \lambda(s)) \in W$.

Since

$$(\sum_{i=1}^{s} \mu(i)t^{a+h(v_i)}g_i^*)(\sum_{i\in[s],b-h(v_i)\in H} \lambda(i)t^{b-h(v_i)}g_i) = (\sum_{i=1}^{s} \mu(i)\lambda(i))t^{a+b},$$

it follows that $(\sum_{i=1}^{s} \mu(i)t^{a+h(v_i)}g_i^*)(C_b) = 0$ if and only if either $a+b \notin H$ or $\langle \mu, \lambda\rangle = \sum_{i=1}^{s} \mu(i)\lambda(i) = 0$ for all $\lambda \in W$ satisfying $\lambda(i) = 0$ for all i with $b - h(v_i) \notin H$. In particular, we have if $\langle \mu, \lambda\rangle = 0$ for all $\lambda \in W$, then $(\sum_{i=1}^{s} \mu(i)t^{a+h(v_i)}g_i^*)(C) = 0$.

For the converse, we assume that $(\sum_{i=1}^{s} \mu(i)t^{a+h(v_i)}g_i^*)(C) = 0$. Write $a = a_+ - a_-$ with $a_+ \in H$ and $a_- \in H$, and set $b_0 = \sum_{i=1}^{s} h(v_i) + a_-$. Since $a + b_0 \in H$ and $b_0 - h(v_i) \in H$ for all $i \in [s]$, and since $(\sum_{i=1}^{s} \mu(i)t^{a+h(v_i)}g_i^*)(C_{b_0}) = 0$, it follows that $(\mu, \lambda) = 0$ for all $\lambda \in W$. Therefore the statement (6) has been proved and this completes the proof. □

Now for any $a \in \mathbb{Z}H$ we want to determine the dimension of $(\operatorname{Im}\delta^*)_a$. We observe that the $\mathbb{Z}H$-graded R-module $\operatorname{Im}\delta^*$ is generated by the elements

$$\delta^*((dx_i)^*) = \sum_{j=1}^{s} (\partial f_{v_j}/\partial x_i)g_j^* = \sum_{j=1}^{s} v_j(i)t^{h(v_j)-h_i}g_j^*.$$

Note that $\deg \delta^*((dx_i)^*) = -h_i$ for $i = 1, \ldots, n$.

For $i = 1, \ldots, n$ we set $w_i = (v_1(i), \ldots, v_s(i))$, and for $a \in \mathbb{Z}H$ we let KD_a be the K-subspace of K^s spanned by the vectors w_i for which $i \in \mathcal{G}_a$. Here the set $\mathcal{G}_a$ is defined to be

$$\mathcal{G}_a = \{i \in [n] \colon\ a + h_i \in H\}.$$

Proposition 1.2. *Let $a \in \mathbb{Z}H$. Then*

$$\dim_K(\operatorname{Im}\delta^*)_a = \dim_K KD_a.$$

Proof. The K-subspace $(\operatorname{Im}\delta^*)_a \subset (F^*)_a$ is spanned by the vectors

$$t^{a+h_i}\delta^*((dx_i)^*) = \sum_{j=1}^{s} v_j(i)t^{a+h(v_j)}g_j^*$$

with $i \in \mathcal{G}_a$.

The desired formula $\dim_K(\operatorname{Im}\delta^*)_a$ follows once we have shown that

$$\sum_{i\in\mathcal{G}_a} \mu(i)t^{a+h_i}\delta^*((dx_i)^*) = 0 \text{ if and only if } \sum_{i\in\mathcal{G}_a} \mu(i)w_i = 0.$$

Here $\mu(i) \in K$ for any $i \in \mathcal{G}_a$. To prove it we notice that

$$\sum_{i\in\mathcal{G}_a} \mu(i)t^{a+h_i}\delta^*((dx_i)^*) = \sum_{i\in\mathcal{G}_a} \mu(i)(\sum_{j=1}^{s} v_j(i)t^{a+h(v_j)}g_j^*) = \sum_{j=1}^{s}(\sum_{i\in\mathcal{G}_a} \mu(i)v_j(i)t^{a+h(v_j)}g_j^*).$$

Thus $\sum_{i\in\mathcal{G}_a} \mu(i)t^{a+h_i}\delta^*((dx_i)^*) = 0$ if and only if $\sum_{i\in\mathcal{G}_a} \mu(i)v_j(i) = 0$ for $j = 1, \ldots, s$. Since $v_j(i) = w_i(j)$, this is the case if and only if $\sum_{i\in\mathcal{G}_a} \mu(i)w_i = 0$. □

Corollary 1.3. *Let $a \in \mathbb{Z}H$. Then $\dim_K KD_a + \dim_K KL_a \leq \dim_K KL$. Equality holds if and only if $T^1(R)_a = 0$.*

Summarizing our discussions of this section we observe that all information which is needed to compute $\dim_K T^1(R)_a$ can be obtained from the $(s \times n)$-matrix

$$A_H = \begin{pmatrix} v_1(1) & v_1(2) & \dots & v_1(n) \\ v_2(1) & v_2(2) & \dots & v_2(n) \\ \vdots & \vdots & & \vdots \\ v_s(1) & v_s(2) & \dots & v_s(n) \end{pmatrix}.$$

Indeed, $\dim_K T^1(K[H])_a$ can be computed as follows: let $l = \operatorname{rank} A_H$, r_a the rank of the submatrix of A_H whose rows are the ith rows of A_H for which $a + h(v_i) \notin H$, and let c_a be the rank of the submatrix of A_H whose columns are the jth columns of A_H for which $a + h_j \in H$. Then

$$\dim_K T^1(K[H])_a = l - l_a - d_a. \tag{7}$$

Corollary 1.4. *Suppose $a \in H$. Then $T^1(R)_a = 0$.*

Proof. Since $a \in H$, it follows that $\mathcal{G}(a) = [n]$ and $\dim_K D_a = \dim_K KL = \operatorname{rank} A_H$. Thus the assertion follows from Corollary 1.3. □

The inequality of Corollary 1.3 can also be deduced from the following lemma.

Lemma 1.5. *Fix $a \in \mathbb{Z}H$. Then $v_i(j) = 0$ for every pair i, j with $i \notin \mathcal{F}_a$ and $j \in \mathcal{G}_a$.*

Proof. Assume on the contrary that $v_i(j) \neq 0$, say $v_i(j) < 0$, for some $i \notin \mathcal{F}_a$ and $j \in \mathcal{G}_a$. Then

$$h(v_i) = -\sum_{\substack{k \\ v_i(k)<0}} v_i(k)h_k = h_j + b, \text{ where } b = \sum_{\substack{k \neq j \\ v_i(k)<0}} -v_i(k)h_k + (-v_i(j) - 1)h_j \in H.$$

Since $j \in \mathcal{G}_a$, we have $a + h_j \in H$ and so $a + h(v_i) = (a + h_j) + b \in H$. Consequently, $i \in \mathcal{F}_a$, a contradiction. □

2. Separable and inseparable saturated lattices

In this section we study conditions under which an affine semigroup ring $K[H]$ is obtained from another affine semigroup ring $K[H']$ by specialization, that is, by reduction modulo a regular element. Of course we can always choose $H' = H \times \mathbb{N}$ in which case $K[H']$ is isomorphic to the polynomial ring $K[H][y]$ over $K[H]$ in the variable y, and $K[H]$ is obtained from $K[H']$ by reduction modulo the regular element y. This trivial case we do not consider as a proper solution of finding an $K[H']$ that specializes to $K[H]$. If no non-trivial $K[H']$ exists, which specializes to $K[H]$, then H will be called inseparable and otherwise separable. It turns out that the separability of H is naturally phrased in terms of the relation lattice L of H.

Let $L \subset \mathbb{Z}^n$ be a subgroup of $\mathbb{Z}^n$. Such a subgroup is often called a lattice. The ideal I_L generated by all binomials f_v with $v \in L$ is called the *lattice ideal* of L. The following properties are known to be equivalent:

(i) $\mathbb{Z}^n/L$ is torsionfree;
(ii) I_L is a prime ideal;
(iii) there exists a semigroup H such that $I_L = I_H$.

A proof of these facts can be found for example in [7]. A lattice L for which $\mathbb{Z}^n/L$ is torsionfree is called a *saturated lattice*.

Let $\varepsilon_1, \dots, \varepsilon_n$ be the canonical basis of $\mathbb{Z}^n$ and $\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1}$ the canonical basis of $\mathbb{Z}^{n+1}$. Let $i \in [n]$. We denote by $\pi_i\colon\ \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ the group homomorphism with $\pi_i(\varepsilon_j) = \varepsilon_j$ for $j = 1, \dots, n$ and $\pi_i(\varepsilon_{n+1}) = \varepsilon_i$. For convenience we denote again by π_i the K-algebra homomorphism $S[x_{n+1}] \to S$ with $\pi_i(x_j) = x_j$ for $j = 1, \dots, n$ and $\pi_i(x_{n+1}) = x_i$.

Definition 2.1. Let $L \subset \mathbb{Z}^n$ be a saturated lattice. We say that L is *i-separable* for some $i \in [n]$, if there exists a saturated lattice $L' \subset \mathbb{Z}^{n+1}$ such that

(i) $\operatorname{rank} L' = \operatorname{rank} L$;
(ii) $\pi_i(I_{L'}) = I_L$;
(iii) there exists a minimal system of generators $f_{w_1}, \dots, f_{w_s}$ of $I_{L'}$ such that the vectors $(w_1(n+1), \dots, w_s(n+1))$ and $(w_1(i), \dots, w_s(i))$ are linearly independent.

The lattice L is called *i-inseparable* if it is not i-separable, and L is called *inseparable* if it is i-inseparable for all i. Moreover, the lattice L' satisfying (i)-(iii) is called an *i-separation lattice* for L. We also call a semigroup H and its toric ring *inseparable* if the relation lattice of H is inseparable.

Remark 2.2. Suppose that $L' \subset \mathbb{Z}^{n+1}$ is an i-separation lattice for L. Let $I_{L'} \subset S[x_{n+1}]$ be the lattice ideal of L'. It is easily seen that $x_{n+1} - x_i \notin I_{L'}$ because $\operatorname{rank} L = \operatorname{rank} L'$. Indeed, if $x_{n+1} - x_i \in I_{L'}$, then $S[x_{n+1}]/I_{L'} \cong S/I_L$, and so $\operatorname{rank} L' = \operatorname{height} I_{L'} = \operatorname{height} I_L + 1 = \operatorname{rank} L + 1$, contradicting Definition 2.1(i). Moreover, $x_{n+1} - x_i$ is a non-zerodivisor of $S[x_{n+1}]/I_{L'}$ since $S[x_{n+1}]/I_{L'}$ is a domain. In particular, if $f_{w_1}, \dots, f_{w_s}$ is a minimal system of generators of $I_{L'}$, then $\pi_i(f_{w_1}), \dots, \pi_i(f_{w_s})$ is a minimal system of generators of I_L, (see Lemma 2.5 for the details). This implies that

$$w_j(i)w_j(n+1) \geq 0 \quad \text{for} \quad j = 1, \dots, s.$$

Indeed, x_i divides $\pi_i(f_{w_j})$ if $w_j(i)w_j(n+1) < 0$. Since $\pi_i(f_{w_j})$ is a minimal generator of I_L and since I_L is a prime ideal, the polynomial $\pi_i(f_{w_j})$ must be irreducible. So, $w_j(i)w_j(n+1) < 0$ is not possible.

Let $v_j = \pi_i(w_j)$ for $j = 1, \dots, s$. Since $w_j(i)w_j(n+1) \geq 0$ for $j = 1, \dots, s$, for all j we have $\pi_i(f_{w_j}) = f_{v_j}$. Hence $f_{v_1} \dots, f_{v_s}$ is a minimal system of generators of I_L.

For an affine semigroup $H \subset \mathbb{Z}^m$ the semigroup ring $K[H]$ is standard graded, if and only if there exists a linear form $\ell = a_1z_1 + a_2z_2 + \cdots + a_mz_m$ in the polynomial ring $\mathbb{Q}[z_1, \dots, z_m]$ such that $\ell(h_i) = 1$ for all minimal generators h_i of H.

The following result provides a necessary condition of i-inseparability. Recall from [6] that an affine semigroup H is called *positive* if $H_0 = \{0\}$, where H_0 is the set of invertible elements of H.

Theorem 2.3. *Let H be a positive affine semigroup which is minimally generated by $h_1, \ldots, h_n$, $L \subset \mathbb{Z}^n$ the relation lattice of H. Suppose that L is i-separable. Then $T^1(K[H])_{-h_i} \neq 0$. In particular, if $K[H]$ is standard graded, then L is inseparable, if $T^1(K[H])_{-1} = 0$.*

Proof. Since L is i-separable, there exists a saturated lattice L' satisfying the conditions (i) and (ii) as given in Definition 2.1. Since $x_{n+1} - x_i$ is a non-zerodivisor on $R' = S[x_{n+1}]/I_{L'}$ it follows that $R'' = R'/(x_{n+1} - x_i)^2 R'$ is an infinitesimal deformation of R (which is isomorphic to $R''/((x_{n+1} - x_i)R'')$.

Let $v_j = \pi_i(w_j)$ for $j = 1, \ldots, s$. By Remark 2.2, we have $\pi_i(f_{w_j}) = f_{v_j}$ for $j = 1, \ldots, s$ and $f_{v_1}, \ldots, f_{v_s}$ is a minimal system of generators of I_L.

Note that $S[x_{n+1}] = S[x_{n+1} - x_i]$. We set ε to be the residue class of $x_{n+1} - x_i$ in $S[x_{n+1} - x_i]/(x_{n+1} - x_i)^2$. Then $S[x_{n+1} - x_i]/(x_{n+1} - x_i)^2 = S[\varepsilon]$. Let $\sigma\colon S[x_{n+1}] \to S[\varepsilon]$ the canonical epimorphism and let J be the image of $I_{L'}$ in $S[\varepsilon]$. Then $R'' = S[\varepsilon]/J$.

In order to determine the generators of J, we fix a j with $1 \leq j \leq s$, and may assume that $w(n+1) \geq 0$ and $w(i) \geq 0$. Then modulo $(x_{n+1} - x_i)^2$, we obtain

$$\begin{aligned} f_{w_j} &= \prod_{\substack{1\leq k\leq n \\ w_j(k)\geq 0}} x_k^{w_j(k)} x_{n+1}^{w_j(n+1)} - \prod_{\substack{1\leq k\leq n \\ w_j(k)<0}} x_k^{w_j(k)} \\ &= \prod_{\substack{1\leq k\leq n \\ w_j(k)\geq 0}} x_k^{w_j(k)} (x_i^{w_j(n+1)} + w_j(n+1) x_i^{w_j(n+1)-1}\varepsilon) - \prod_{\substack{1\leq k\leq n \\ w_j(k)<0}} x_k^{w_j(k)} \\ &= f_{v_j} + [w_j(n+1)(\prod_{\substack{1\leq k\leq n \\ v_j(k)\geq 0}} x_k^{v_j(k)})/x_i]\varepsilon. \end{aligned}$$

For the second equality we used that $x_{n+1} = \varepsilon + x_i$ and $\varepsilon^2 = 0$, and the third equality is due to the fact $v_j(i) = w_j(i) + w_j(n+1)$.

The homomorphism $\varphi : I_L/I_L^2 \to R$ corresponding to the infinitesimal deformation $S[\varepsilon]/J$ is given by

$$\varphi(f_{v_j} + I_L^2) = w_j(n+1)(\prod_{\substack{1\leq k\leq n \\ v_j(k)\geq 0}} x_k^{v_j(k)})/x_i + I_L = w_j(n+1)t^{h(v_j)-h_i} \text{ for } j = 1, \ldots, s,$$

which induces the element

$$\alpha = \sum_{1\leq j\leq s} w_j(n+1)t^{h(v_j)-h_i} g_j^* \in (U^*)_{-h_i}.$$

Since H is positive it follows that $\mathcal{G}_{-h_i} = \{i\}$, and this implies that $(\mathrm{Im}\delta^*)_{-h_i} = K\sum_{1\leq j\leq s} v_j(i)t^{h(v_j)-h_i} g_j^*$, see Proposition 1.2. Assume $\alpha \in (\mathrm{Im}\delta^*)_{-h_i}$. Then there exists $\lambda \in K$ such that

$$(w_1(n+1), \ldots, w_s(n+1)) = \lambda(v_1(i), \ldots, v_s(i)).$$

Since $v_j(i) = w_j(i) + w_j(n+1)$ for $j = 1, \ldots, s$, and since by condition (iii) of Definition 2.1 the vectors $(w_1(n+1), \ldots, w_s(n+1)$ and $(w_1(i), \ldots, w_s(i))$ are linearly independent, we obtain a contradiction. Hence $T^1(R)_{-h_i} \neq 0$, as required. $\square$

As a first example of a separable lattice we consider the relation lattice of a numerical semigroup.

Discussion 2.4. Let $H \subset \mathbb{N}$ be the numerical semigroup minimally generated by h_1, h_2, h_3 with $\gcd(h_1, h_2, h_3) = 1$. Recall some facts from [8]. For $i = 1, 2, 3$ let c_i be the smallest integer such that $c_i h_i \in \mathbb{N}h_k + \mathbb{N}h_\ell$, where $\{i, k, \ell\} = [3]$, and let r_{ik} and $r_{i\ell}$ be nonnegative integers such that $c_i h_i = r_{ik}h_k + r_{i\ell}h_\ell$. Denote by L the relation lattice of H. Then the three vectors

$$v_1 = (c_1, -r_{12}, -r_{13}), \quad v_2 = (-r_{21}, c_2, -r_{23}), \quad v_3 = (-r_{31}, -r_{32}, c_3)$$

generate L. We have $v_1 + v_2 + v_3 = 0$ if

(1) all $r_{ij} \neq 0$, or
(2) $v_1 = (c_1, -c_2, 0)$, $v_2 = (0, c_2, -c_3)$ and $v_3 = (-c_1, 0, c_3)$.

In case (1), $f_{v_1}, f_{v_2}, f_{v_3}$ is the unique minimal system of generators of I_L. In case (2), $f_{v_1} + f_{v_2} + f_{v_3} = 0$, so that any two of the f_{v_i}'s minimally generate I_L.

An example for (1) is the semigroup with generators 3, 4 and 5, and example for (2) is the semigroup with generators 6, 10 and 15. (3) If $v_1 + v_2 + v_3 \neq 0$, then there exist distinct integers $k, \ell \in [3]$ such that $v_k + v_\ell = 0$ and $r_{ij} \neq 0$ for $i \in [3] \setminus \{k, \ell\}$ and $j \in \{k, \ell\}$. In this case I_L is minimally generated by $x_i^{c_i} - x_k^{r_{ik}} x_l^{r_{il}}$ and $x_k^{c_k} - x_\ell^{c_\ell}$.

An example for (3) is the semigroup with generators 4, 5 and 6.

It is known and easy to prove that $R = K[H]$ is not rigid. Indeed, since R is quasi-homogeneous, the Euler relations $\sum_{i=1}^n (\partial f/\partial x_i)x_i = (\deg f)f$ imply that there is an epimorphism $\chi \colon \Omega_{R/K} \to \mathfrak{m}$ with $\chi(dx_i) \mapsto t^{h_i}$ where $\mathfrak{m} = (t^{h_1}, t^{h_2}, t^{h_3})$ is the graded maximal ideal of R. Since $\operatorname{rank} \Omega_{R/K} = \operatorname{rank} \mathfrak{m} = 1$, it follows that $C = \operatorname{Ker} \chi$ is a torsion module. Thus we obtain the following exact sequence

$$0 \to C \to \Omega_{R/K} \to \mathfrak{m} \to 0,$$

which induces the long exact sequence

$$\operatorname{Hom}_R(C, R) \to \operatorname{Ext}^1_R(\mathfrak{m}, R) \to \operatorname{Ext}^1_R(\Omega_{R/K}, R).$$

Since R is a 1-dimensional domain, R is Cohen-Macaulay, $\operatorname{Hom}_R(C, R) = 0$ and $\operatorname{Ext}^1_R(\mathfrak{m}, R) \cong \mathfrak{m}^{-1}/R \neq 0$. It follows that $\operatorname{Ext}^1_R(\Omega_{R/K}, R) \neq 0$. In other words, R is not rigid.

Of course the same argument can be applied to any numerical semigroup generated by more than 1 element.

We have seen that $K[H]$ is not rigid. The next result shows that the relation lattice of H is even i-separable for $i \in [3]$ with $T^1(R)_{-h_i} \neq 0$. To prove this we need

Lemma 2.5. *Let $L \subset \mathbb{Z}^n$ and $L' \subset \mathbb{Z}^{n+1}$ be saturated lattices which satisfy the conditions* (i) and (ii) *as given in Definition 2.1. Then*

(a) *I_L and $I_{L'}$ have the same number of minimal generators;*

(b) $f_{w_1}, \ldots, f_{w_s}$ *is a minimal system of generators of* $I_{L'}$ *if and only if* $\pi_i(f_{w_1}), \ldots, \pi_i(f_{w_s})$ *is a minimal system of generators of* I_L.

Proof. (a) For any $S[x_{n+1}]/I_{L'}$-module M we denote by $\overline{M}$ its reduction modulo $x_{n+1} - x_i$. The conditions (i) and (ii) of Definition 2.1 guarantee that $x_{n+1} - x_i$ is a non-zerodivisor on $S[x_{n+1}]/I_{L'}$ and that $S/I_L \cong \overline{S[x_{n+1}]/I_{L'}}$. From these facts (a) follows.

(b) Suppose that $f_{w_1}, \ldots, f_{w_s}$ is a minimal system of generators of $I_{L'}$. Then I_L is generated by $\pi_i(f_{w_1}), \ldots, \pi_i(f_{w_s})$ since $\pi_i(I'_L) = I_L$. By (a), $\pi_i(f_{w_1}), \ldots, \pi_i(f_{w_s})$ is a minimal system of generators of I_L.

Conversely, assume that $\pi_i(f_{w_1}), \ldots, \pi_i(f_{w_s})$ is a minimal system of generators of I_L. We want to show that $I_{L'} = (f_{w_1}, \ldots, f_{w_s})$. Set $J = (f_{w_1}, \ldots, f_{w_s})$. Then we obtain the following short exact sequence:

$$0 \to I_{L'}/J \to S[x_{n+1}]/J \xrightarrow{\alpha} S[x_{n+1}]/I_{L'} \to 0$$

Here α is the natural epimorphism. By [6, Proposition 1.1.4], we obtain the exact sequence

$$0 \to \overline{I_{L'}/J} \to \overline{S[x_{n+1}]/J} \xrightarrow{\overline{\alpha}} \overline{S[x_{n+1}]/I_{L'}} \to 0.$$

Since $\pi_i(J) = \pi_i(I'_L) = I_L$ it follows that $\overline{\alpha}$ is an isomorphism, and so $\overline{I_{L'}/J} = 0$. Nakayama's Lemma implies that $I_{L'}/J = 0$. Hence $J = I_{L'}$, as desired. □

Proposition 2.6. *Let H be a numerical semigroup as above and set $R = K[H]$. Let $L \subset \mathbb{Z}^3$ be the relation lattice of H. With the notation of Discussion 2.4 we have:*

(a) *If $v_1 + v_2 + v_3 = 0$, then $\dim_K T^1(R)_{-h_i} = 1$ and L is i-separable for $i = 1, 2, 3$.*
(b) *If $v_1 + v_2 + v_3 \neq 0$, then there exists $i \in [3]$ such that $I_L = (x_i^{c_i} - x_k^{r_{ik}} x_l^{r_{il}}, x_k^{c_\ell} - x_\ell^{c_k})$ with $\{i, k, l\} = [3]$ and $r_{ik}, r_{il} \neq 0$. In this case, $T^1(R)_{-h_i} = 0$, and for $j \neq i$ we have that $T^1(R)_{-h_j} \neq 0$ and that L is j-separable.*

Proof. (a) We consider the case (1), where $r_{ij} > 0$ for all i and j, see Discussion 2.4. Fix $i \in [3]$. Since all $r_{ij} > 0$ it follows that $\mathcal{F}_{-h_i} = \{1, 2, 3\}$, and since H is a positive semigroup we have $\mathcal{G}_{-h_i} = \{i\}$. It follows from Corollary 1.1 and Proposition 1.2 that $\dim_K (U^*)_{-h_i} = 2$ and $\dim_K (\mathrm{Im}(\delta^*_{-h_i}) = 1$. Hence $\dim_K T^1(R)_{-h_i} = 1$.

Consider the vectors

$$\begin{aligned} w_1 &= (c_1 - 1, -r_{12}, -r_{13}, 1), \\ w_2 &= (-r_{21} + 1, c_2, -r_{23}, -1), \\ w_3 &= (-r_{31}, -r_{32}, c_3, 0) \end{aligned}$$

in $\mathbb{Z}^4$, and set $L' = \mathbb{Z}w_1 + \mathbb{Z}w_2 + \mathbb{Z}w_3$. We will prove that L' is a 1-separation of L. First we show that L' is saturated. Indeed, if $aw \in L'$ for some $0 \neq a \in \mathbb{Z}$ and some $w \in \mathbb{Z}^4$, then $aw = a_1 w_1 + a_2 w_2 + a_3 w_s$ for some $a_i \in \mathbb{Z}$, and it follows that $av = a_1 v_1 + a_2 v_2 + a_3 v_3$, where $v = \pi_1(w)$. This implies that $v = k_1 v_1 + k_2 v_2 + k_3 v_3$ for some $k_i \in \mathbb{Z}$, since L is saturated. Thus $(a_1 - ak_1)v_1 + (a_2 - ak_2)v_2 + (a_3 - ak_3)v_3 = 0$ and so $a_1 - ak_1 = a_2 - ak_2 = a_3 - ak_3$. It follows that $(a_1 - ak_1)w_1 + (a_2 - ak_2)w_2 + (a_3 - ak_3)w_3 = 0$. Thus $w = k_1 w_1 + k_2 w_2 + k_3 w_3$. Hence L' is saturated. Next we show $\pi_1(I_{L'}) = I_L$. It is clear that $I_L \subseteq \pi_1(I_{L'})$ since $\pi_1(f_{w_i}) = f_{v_i}$ for $i = 1, 2, 3$.

For the converse direction, we only need to note that $\pi_1(L') = L$ and that $f_{\pi_1(w)}$ divides $\pi_1(f_w)$ for all $w \in L'$.

Now, applying Lemma 2.5 we conclude that $f_{w_1}, f_{w_2}, f_{w_3}$ is a minimal system of generators of $I_{L'}$ satisfying condition (iii) of Definition 2.1. Consequently, L' is a 1-separation of L. Similar arguments work for $i = 2, 3$.

In case (2), L is generated by any two of the vectors $v_1 = (c_1, -c_2, 0)$, $v_2 = (0, c_2, -c_3)$ and $v_3 = (-c_1, 0, c_3)$. Let $L' \subset \mathbb{Z}^4$ be a lattice generated by $w_1 = (c_1 - 1, c_2, 0, 1)$ and $w_2 = (-c_1, 0, c_3, 0)$. We claim that L' is a 1-separation of L. Indeed, the ideal of 2-minors $I_2(W)$ of the matrix W whose row vectors are w_1 and w_2 contains the elements c_1 and c_3. By the choice of the c_i's it follows that $\gcd(c_1, c_3) = 1$. Thus, $I_2(W) = \mathbb{Z}$. This shows that L' is saturated. Since $\pi_1(f_{w_i}) = f_{v_i}$ for $i = 1, 2$ and since $I_L = (f_{v_1}, f_{v_2})$, Lemma 2.5 implies that $I_{L'} = (f_{w_1}, f_{w_2})$ and $\pi_1(I_{L'}) = I_L$. Since L' satisfies also condition (iii) of Definition 2.1, it follows that L is 1-separable. In the same way it is shown that L is i-separable for $i = 2, 3$.

(b) This is case (3) of Discussion 2.4 and we have $I_H = (x_i^{c_i} - x_k^{r_{ik}} x_l^{r_{kl}}, x_k^{c_k} - x_\ell^{c_\ell})$ with $\{i, k, l\} = [3]$. Thus I_H is a complete intersection and the exponents c_1, c_2 and c_3 are all > 1. Without loss of generality we may assume that $i = 2$, $k = 1$ and $l = 3$. Since the lattice $L \subset \mathbb{Z}^3$ with basis $v_1 = (-r_{21}, c_2, -r_{23})$, $v_2 = (-c_3, 0, c_1)$ is saturated, it follows that the ideal of 2-minors $(c_1c_2, c_2c_3, c_1r_{21} + c_3r_{23})$ of

$$\begin{pmatrix} -r_{21} & c_2 & -r_{23} \\ -c_3 & 0 & c_1 \end{pmatrix}$$

is equal to $\mathbb{Z}$.

Consider the lattice $L' \subset \mathbb{Z}^4$ whose basis w_1, w_2 consists of the row vectors of

$$\begin{pmatrix} -r_{21} + 1 & c_2 & r_{23} & -1 \\ -c_3 & 0 & c_1 & 0 \end{pmatrix}.$$

The ideal of 2-minors of this matrix contains $(c_1c_2, c_2c_3, c_1r_{21} + c_3r_{23})$, and hence is again equal to $\mathbb{Z}$. Thus L' is saturated. Furthermore we have $\pi_2(f_{w_1}) = f_{v_1}$, $\pi_2(f_{w_2}) = f_{v_2}$ and $\pi_2(L') = L$. This implies that $\pi_2(I_{L'}) = I_L$. Since $\operatorname{rank} L' = \operatorname{rank} L = 2$, the conditions (i) and (ii) of Definition 2.1 are satisfied. Applying Lemma 2.5 we obtain $I_{L'} = (f_{w_1}, f_{w_2})$. Since the condition (iii) of Definition 2.1 is also satisfied we see that L is 1-separable. Similarly, one shows that L is 3-separable.

□

3. Edge rings of bipartite graphs

Let G be a finite simple graph on the vertex set $[m]$, and let K a field. The K-algebra $R = K[G] = K[t_it_j\colon\ \{i,j\} \in E(G)]$ is called the *edge ring* of G. Here $E(G)$ denotes the set of edges of G. We let $n = |E(G)|$, and denote by S the polynomial ring over K in the indeterminates x_e with $e \in E(G)$. Let $\varphi\colon S \to K[G]$ be the K-algebra homomorphism with $x_e \mapsto t_it_j$ for $e = \{i,j\}$. The toric ideal $\operatorname{Ker} \varphi$ will be denoted by I_G.

In this section we will discuss inseparability, semi-rigidity and rigidity of the edge ring of a bipartite graph, which may as well be considered as the toric ring associated with the affine semigroup H generated by the elements $\delta_i + \delta_j$ with $\{i,j\} \in E(G)$,

where $\delta_1, \dots, \delta_m$ is a canonical basis of $\mathbb{Z}^m$. Let G be a bipartite graph. The generators of I_G are given in terms of even cycles of G. Recall that a walk in G is a sequence $C\colon i_0, i_1, \dots, i_q$ such that $\{i_k, i_{k+1}\}$ is an edge of G for $k = 0, 1, \dots, k-1$. C is called a closed walk, if $i_q = i_0$. The closed walk C is called a cycle if $i_j \neq i_k$ for all $j \neq k$ with $j, k < q$, and it is called an even closed walk if q is even. Observe that any cycle of bipartite graph is an even cycle.

Given any even cycle (more generally an even closed walk) $C\colon i_0, i_1, \dots, i_{2q}$. The edges of C are $e_{k_j} = \{i_j, i_{j+1}\}$ for $j = 0, 1, \dots, 2q-1$ together with the edge $e_{k_{2q-1}} = \{i_{2q-1}, i_0\}$. We associate to C the vector $v(C) \in \mathbb{Z}^n$ which defined as

$$v(C) = \sum_{i=0}^{q-1} \varepsilon_{k_{2i}} - \sum_{i=0}^{q-1} \varepsilon_{k_{2i+1}} \tag{8}$$

Here $\varepsilon_1, \dots, \varepsilon_n$ denotes the canonical basis of $\mathbb{Z}^n$. Note that $v(C)$ is determined by C only up to sign. We call $v(C)$ as well as $-v(C)$ the vector corresponding to C.

For simplicity we write f_C for $f_{v(C)}$. Recall from [11] that the toric ideal I_G of a finite bipartite graph is minimally generated by indispensable binomials, that is, by binomials, which up to sign, belong to any system of generators of I_G. Furthermore, a binomial $f \in I_G$ is indispensable if and only if $f = f_C$, where C is an induced cycle, that is, a cycle without a chord. In particular, if G' is the graph obtained from G by deleting all edges which do not belong to any cycle, then $I_G = I_{G'}S$. Therefore we may assume throughout this section that each edge of G belongs to some cycle.

Now for the rest of this section we let G be a bipartite graph on the vertex set $[m]$ with edge set $E(G) = \{e_1, \dots, e_n\}$. With the edge $e_k = \{i, j\}$ we associate the vector $h_k = \delta_i + \delta_j$. Here $\delta_1, \dots, \delta_m$ is the canonical basis of $\mathbb{Z}^m$. The semigroup generated by $h_1, \dots, h_n$ we denote by $H(G)$ or simply by H. Note that $K[H(G)] = K[G]$.

Let $\{C_1, \dots, C_s\}$ be the set of cycles of G and $v_i = v(C_i)$ the vector corresponding to C_i. We may assume that for $i = 1, \dots, s_1 \leq s$, the cycles C_i are the induced cycles of G. Then I_G is minimally generated by $f_{v_1}, \dots, f_{v_{s_1}}$, see [11]. Of course, I_G is also generated by $f_{v_1}, \dots, f_{v_s}$. In particular, if L is the relation lattice of H, then KL is the vector space spanned by $v_1, \dots, v_s$.

Let $a \in \mathbb{Z}H$. As in Section 1 we set

$$\mathcal{F}_a = \{1 \leq i \leq s\colon\ a + h(v_i) \in H\}, \text{ and } KL_a = \mathrm{Span}_K\{v_i\colon\ i \in [s] \setminus \mathcal{F}_a\}.$$

In addition we now also set

$$\mathcal{F}'_a = \{1 \leq i \leq s_1\colon\ a + h(v_i) \in H\}, \text{ and } KL'_a = \mathrm{Span}_K\{v_i\colon\ i \in [s_1] \setminus \mathcal{F}'_a\}.$$

In general, $\mathcal{F}'_a$ is a proper subset of $\mathcal{F}_a$. However, we have

Lemma 3.1. *$KL'_a = KL_a$ for all $a \in \mathbb{Z}H$.*

Proof. Since $[s_1] \setminus \mathcal{F}'_a \subseteq [s] \setminus \mathcal{F}_a$, we have $KL'_a \subseteq KL_a$. Let $i \in ([s] \setminus \mathcal{F}_a) \setminus ([s_1] \setminus \mathcal{F}'_a)$. Then $a + h(v_i) \notin H$ and C_i is a cycle with chords. In the following we describe a process to obtain the induced cycles with vertex set contained in $V(C_i)$. Choose a chord of C_i and note that this chord divides C_i into two cycles. If both cycles are

induced, then the process stops. Otherwise we divide as before, those cycles which are not induced. Proceeding in this way, we obtain induced cycles of G, denoted by $C_{i_1}, \dots, C_{i_k}$, such that $E(C_{i_j})$ consists of at least one chord of C_i. Moreover, the edges of C_{i_j} which are not chords of C_i, are edges of C_i.

In general, if C is a cycle and $v = v(C)$, then

$$h(v) = \sum_{j \in V(C)} \delta_j.$$

Hence it follows from the construction of the induced cycles C_{i_j} that $h(v_i) - h(v_{i_j})$ is the sum of certain terms $\delta_{k_1} + \delta_{k_2}$, where $\{k_1, k_2\}$ is an edge of C_i, and hence $h(v_i) - h(v_{i_j}) \in H$. Since $a + h(v_i) \notin H$ it follows that $a + h(v_{i_j}) \notin H$ for all j. This implies that $v_{i_j} \in KL'_a$ for all j, and so $v_i \in KL'_a$ since v_i is a linear combination of the v_{i_j}. □

For the discussion on separability we need to know when $T^1(K[G])_{-h_j}$ vanishes, see Theorem 2.3. For that we need to have the interpretation of $\mathcal{F}_{-h_j}$ for edge rings which is given by the following formula:

$$\mathcal{F}_{-h_j} = \{i \in [n] : V(e_j) \subset V(C_i)\}. \tag{9}$$

For the proof of this equation note that if $V(e_j) \subseteq V(C_i)$, then without loss of generality we assume that $C_i : 1, 2, \dots, 2t$ and that $e_j = \{1, k\}$ with $k \in [2t]$. Note that k is even, since G contains no odd cycle. It follows that $-h_j + h(v_i) = (\delta_2 + \delta_3) + \cdots + (\delta_{k-2} + \delta_{k-1}) + (\delta_{k+1} + \delta_{k+2}) + \cdots + (\delta_{2t-1} + \delta_{2t}) \in H$, and so $i \in \mathcal{F}_{-h_j}$ by definition. Conversely, assume that $V(e_j) \nsubseteq V(C_i)$ and let $k \in V(e_j) \setminus V(C_i)$. Then $-h_j + h(v_i)$ is a vector in $\mathbb{Z}^m$ with the kth entry negative and thus it does not belong to H. Therefore $i \notin \mathcal{F}_{-h_j}$.

Later we also shall need

Lemma 3.2. *Let $W\colon i_1, i_2, \dots, i_{2k}, i_1$ be an even closed walk in G and let e_j be an edge of G with the property that $e_j \neq \{i_a, i_b\}$ with $1 \le a < b \le 2k$. Then the vector $w = v(W) \in KL$ belongs to KL_{-h_j}.*

Proof. We may view W as a bipartite graph with bipartition $\{i_1, i_3, \dots, i_{2k-1}\}$ and $\{i_2, i_4, \dots, i_{2k}\}$. Then we see that w belongs to the space spanned by the vectors corresponding to the induced cycles of G with edges in W. This vector space is a subspace of KL_{-h_j}, since e_j is not an edge of any cycle with edges in W, as follows from (9). □

We call the space KL which is spanned by the vectors $v_1, \dots, v_s$ the *cycle space* of G (with respect to K). Usually the cycle space is only defined over $\mathbb{Z}_2$. For bipartite graphs the dimension of the cycle space does not depend on K and is known to be

$$|E(G)| - |V(G)| + c(G). \tag{10}$$

where $c(G)$ is the number of connected components of G, see [13, Corollary 8.2.13].

Inseparability. In this subsection we present some characterizations of bipartite graphs G for which $K[G]$ is inseparable.

Note that (9) says that $i \in \mathcal{F}_{-h_j}$ if and only if e_j is an edge or a chord of C_i. Accordingly, we split the set $\mathcal{F}_{-h_j}$ into the two subsets

$$\mathcal{A}_j = \{i \in [s]\colon\ e_j \text{ is an edge of } C_i\}, \tag{11}$$

and

$$\mathcal{B}_j = \{i \in [s]\colon\ e_j \text{ is a chord of } C_i\}. \tag{12}$$

We also set $V_{-h_j} = \operatorname{Span}_K\{v_i : i \in [s] \setminus \mathcal{A}_j\}$. Then, since by assumption all edges of G belong to a cycle, we obtain

$$\dim_K V_{-h_j} = \dim_K KL - 1 \quad \text{for} \quad j = 1, \ldots, n. \tag{13}$$

Indeed, let $G \setminus \{e_j\}$ be the graph obtained from G by deleting the edge e_j and leaving vertices unchanged. Then V_{-h_j} is the cycle space of $G \setminus \{e_j\}$.

Lemma 3.3. *$T^1(R)_{-h_j} = 0$ if and only if for all $i \in \mathcal{B}_j$, one has $v_i \in KL_{-h_j}$.*

Proof. Since $-h_j + h_i \in H$ if and only if $i = j$ it follows that $\dim(\operatorname{Im}\delta^*)_{-h_j} = 1$, see Proposition 1.2. Thus, since $KL_{-h_j} \subseteq V_{-h_j}$, it follows from (13) that $T^1(R)_{-h_j} = 0$ if and only if $V_{-h_j} = KL_{-h_j}$. Since $V_{-h_j} = KL_{-h_j} + \operatorname{Span}_K\{v_i : i \in \mathcal{B}_j\}$, the assertion follows. □

For stating the next result we have first to introduce some concepts. Let C be a cycle. Then the path $P\colon i_1, i_2, i_3, \ldots, i_{r-1}, i_r$ (with $r \geq 2$ and with $i_j \neq i_k$ for all $j \neq k$) is called a *path chord* of C if $i_1, i_r \in V(C)$ and $i_j \notin V(C)$ for all $j \neq i_1, i_r$. The vertices i_1 and i_r are called the ends of P. Note that any chord of C is a path chord.

Let P be a path chord of C. We may assume that $\{i, i+1\}$ for $i = 1, \ldots, t$ together with $\{1, 2t\}$ are the edges of C and that $i_1 = 1$ and $i_r = k$ with $k \neq 1$. Let P' be another path chord of C. Then we say that P and P' *cross* each other if one end of P' belongs to the interval $[2, k-1]$ and the other end of P' belongs to $[k+1, 2t]$. In particular, if P is a chord and P' crosses P, we say that P' is a *crossing path chord* of C with respect to the chord P.

Theorem 3.4. *Let G be a bipartite graph with edge set $\{e_1, \ldots, e_n\}$, and let $R = K[G]$ be the edge ring of G. Then the following conditions are equivalent:*

(a) $T^1(R)_{-h_j} \neq 0$.
(b) *There exists a cycle C of G for which e_j is a chord, and there is no crossing path chord P of C with respect to e_j.*
(c) *The relation lattice of $H(G)$ is j-separable.*

Proof. (a) ⇒ (b): Assume that (b) does not hold. Let $i \in \mathcal{B}_j$ with $\mathcal{B}_j$ as defined in 12. By our assumption, C_i admits a path chord, denoted by P, which crosses e_j. Denote by i_1, i_2 the two ends of P. Then C is the union of two paths P_1 and P_2 which both have ends i_1, i_2. Since $P_1 \cup P$ and $P_2 \cup P$ are cycles and e_j is neither an edge nor a chord of them, it follows from Lemma 3.2 that the vectors $w_1 = v(P_1 \cup P)$ and $w_2 = v(P_2 \cup P)$ belong to KL_{-h_j}. Therefore, $v_i \in KL_{-h_j}$ because it is a linear combination of w_1 and w_2. Now applying Lemma 3.3, we obtain $T^1(R)_{-h_j} = 0$, a contradiction.

(b) $\Rightarrow$ (c): We may assume that the cycle C as given in (b) has the edge set

$$E(C) = \{e_1 = \{1,2\}, \ldots, e_\ell = \{\ell, \ell+1\}, \ldots, e_{2t} = \{2t, 1\}\},$$

and that $e_j = \{1, k\}$ with $2 < k < 2t - 1$.

We let X be the set of all $a \in [m] \setminus V(C)$ for which there is a path P from a to some vertex of $[2, k-1]$, and we set $Y = [m] \setminus (V(C) \cup X)$.

We now define a graph $G' = G_1 \cup G_2$, where G_1 and G_2 are disjoint graphs, that is, $V(G_1) \cap V(G_2) = \emptyset$. The graph G_2 is the subgraph of G induced on $X \cup [k]$. Next we first define $\widetilde{G}_1$ as the subgraph of G induced on $Y \cup [k+1, 2t] \cup \{1, k\}$. Then G_1 is obtained from $\widetilde{G}_1$ by renaming 1 as $m+1$ and k as $m+2$. We claim that G_1 and G_2 are disjoint. Indeed, $V(G_1) \cap V(G_2) \subseteq [k+1, 2t] \cap X$. Condition (b) implies that $[k+1, 2t] \cap X = \emptyset$.

Now we claim that if we identify in G' the vertex $m+1$ with 1 and the vertex $m+2$ with k, then we obtain G. Indeed, let G'' be the graph which is obtained from G' after this identification. We have to show that $G'' = G$. Obviously, we have $V(G'') = V(G)$ and $E(G'') \subseteq E(G)$. Let $e \in E(G) \setminus E(G'')$. Then $e = \{k_1, k_2\}$ with $k_1 \in [k+1, 2t] \cup Y$ and $k_2 \in X \cup [2, k-1]$. If $k_2 \in [2, k-1]$, then $k_1 \in [1, k] \cap X$ by the definition of X. This is impossible since $(X \cup [1, k]) \cap ([k+1, 2t] \cup Y) = \emptyset$; If $k_2 \in X$, then again by the definition of X it follows that $k_1 \in X \cup [1, k]$, which is impossible again in the same reason. Thus we have proved the claim.

Now the edge ring of G' is of the form $R' = S'/I_{G'} = S'/(I_{G_1} + I_{G_2})S'$, where $S' = S[x_{n+1}]$ and where the variable x_{n+1} corresponds to the edge $e_{n+1} = \{m+1, m+2\}$. The variable x_j corresponds to the edge e_j if $e_j \in G$, and to $e \in E(G_1)$ if $e \in E(G_1) \setminus \{m+1, m+2\}$ and e is mapped to e_j by the identification map $G' \to G'' = G$. Let L be the relation lattice of $H(G)$ and L' be the relation lattice of $H(G')$. Then $L \subset \mathbb{Z}^n$ and $L' \subset \mathbb{Z}^{n+1}$ are saturated lattices. We claim that L and L' satisfy the conditions (i), (ii) and (iii) with respect to π_j, see Definition 2.1. We first show that $\pi_j(I_{L'}) = I_L$. Let f be a minimal generator of I_L. Then there exists an induced cycle D of G such that $f = f_D$. Since $G = G''$ it follows that $V(D) \subset V(\widetilde{G}_1)$ or $V(D) \subset V(G_2)$. Hence there is an induced cycle D' in G' whose image under the identification map is D. Therefore, $\pi_j(f_{D'}) = f_D$. This proves the condition (ii). Since (ii) is satisfied, it follows that $R'/(x_{n+1} - x_j)R' \cong R$. Moreover, $x_{n+1} - x_j$ is a non-zerodivisor on R', since R' is a domain. This implies that height $I_{L'} =$ height I_L. In particular, $\operatorname{rank} L' = \operatorname{rank} L$. Thus the condition (i) is also satisfied. Finally, by the definition of G_1 and G_2, there exist an induced cycle of G_1 with e_{n+1} as an edge, say C_1, and an induced cycle of G_2 with e_j as an edge, say C_2. Let $w_1 = v(C_1)$ and $w_2 = v(C_2)$. Then $w_1(n+1) \neq 0$, $w_1(j) = 0$, $w_2(n+1) = 0$ and $w_2(j) \neq 0$. This implies the condition (iii).

The implication (c) $\Rightarrow$ (a) follows from Theorem 2.3. $\square$

Corollary 3.5. *Let G be a bipartite graph. Then $K[G]$ is inseparable if and only for any cycle C which has a unique chord e, there exists a crossing path chord of C with respect to e. In particular, if no cycle of G has chord, then G is inseparable.*

Proof. By Theorem 3.4, G is inseparable if and only if for any cycle C with a chord e, there exists a crossing path chord of C with respect to e. Assume first that $K[G]$ is inseparable. Then, by what we just said, each cycle of G with a chord has the desired property. Conversely, assume that each cycle with a unique chord has a crossing path chord, and let C be a cycle with a chord e. Suppose C has another chord, say e'. If e' crosses e, then we are done. Otherwise e' divides C into two smaller cycles C_1 and C_2, and we may assume that e is a chord of C_1. Since C_1 has less chords than C, we may apply induction on the number of chords of a cycle and deduce that there exists a crossing path chord of C_1 with respect to e. Then this path chord is also a crossing path chord of C with respect to e. $\square$

As an example of the theory which we developed so far we consider coordinate rings of convex polyominoes. First we recall from [9] the definitions and some facts about convex polyominoes.

Let $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R} : x, y \geq 0\}$. We consider $(\mathbb{R}_+, \leq)$ as a partially ordered set with $(x, y) \leq (z, w)$ if $x \leq z$ and $y \leq w$. Let $a, b \in \mathbb{Z}_+^2$. Then the set $[a, b] = \{c \in \mathbb{Z}_+^2 : a \leq c \leq b\}$ is called an *interval*.

A *cell* C is an interval of the form $[a, b]$, where $b = a + (1, 1)$. The elements of C are called *vertices* of C. We denote the set of vertices of C by $V(C)$. The intervals $[a, a + (1, 0)], [a + (1, 0), a + (1, 1)], [a + (0, 1), a + (1, 1)]$ and $[a, a + (0, 1)]$ are called *edges* of C. The set of edges of C is denoted by $E(C)$.

Let P be a finite collection of cells of $\mathbb{Z}_+^2$. Then two cells C and D are called *connected* if there exists a sequence $\mathcal{C} : C = C_1, C_2, \ldots, C_t = D$ of cells of $\mathcal{P}$ such that for all $i = 1, \ldots, t-1$ the cells C_i and C_{i+1} intersect in an edge. If the cells in $\mathcal{C}$ are pairwise distinct, then $\mathcal{C}$ is called a *path* between C and D. A finite collection of cells $\mathcal{P}$ is called a *polyomino* if every two cells of $\mathcal{P}$ are connected. The vertex set of $\mathcal{P}$, denoted $V(\mathcal{P})$, is defined to be $\bigcup_{C \in \mathcal{P}} V(C)$ and the edge set of $\mathcal{P}$, denoted $E(\mathcal{P})$, is defined to be $\bigcup_{C \in \mathcal{P}} E(C)$. A polyomino is said to be *vertically* or *column convex* if its intersection with any vertical line is convex. Similarly, a polyomino is said to be *horizontally* or *row convex* if its intersection with any horizontal line is convex. A polyomino is said to be *convex* if it is row and column convex. Figure 1 shows two polyominos whose cells are marked by gray color The right hand side polyomino is convex while the left one is not.

FIGURE 1.

Let $\mathcal{P}$ be a polyomino, and let K be a field. We denote by S the polynomial over K with variables x_{ij} with $(i,j) \in V(\mathcal{P})$. A 2-minor $x_{ij}x_{kl} - x_{il}x_{kj} \in S$ with $i < k$ and $j < l$ is called an *inner minor* of $\mathcal{P}$ if all the cells $[(r,s),(r+1,s+1)]$ with $i \le r \le k-1$ and $j \le s \le l-1$ belong to $\mathcal{P}$. The ideal $I_{\mathcal{P}} \subset S$ generated by all inner minors of $\mathcal{P}$ is called the *polyomino ideal* of $\mathcal{P}$. We also set $K[\mathcal{P}] = S/I_{\mathcal{P}}$. It has been shown in [9] that $K[\mathcal{P}]$ is a domain, and hence a toric ring, if $\mathcal{P}$ is convex. A toric parametrization of $K[\mathcal{P}]$ will be given in the following proof.

Theorem 3.6. *Let $\mathcal{P}$ be a convex polyomino. Then $k[\mathcal{P}]$ is inseparable.*

Proof. Set $A_{\mathcal{P}} = \{h_i : (i,j) \in V(\mathcal{P}) \text{ for some } j \in \mathbb{Z}_+\}$ and $B_{\mathcal{P}} = \{v_j : (i,j) \in V(\mathcal{P}) \text{ for some } i \in \mathbb{Z}_+\}$. We associate with $\mathcal{P}$ a bipartite graph $G(\mathcal{P})$ such that $V(G(\mathcal{P})) = A_P \cup B_{\mathcal{P}}$ and $E(G(\mathcal{P})) = \{\{h_i, v_j\} : (i,j) \in V(\mathcal{P})\}$. Figure 2 shows a polyomino and its associated bipartite graph.

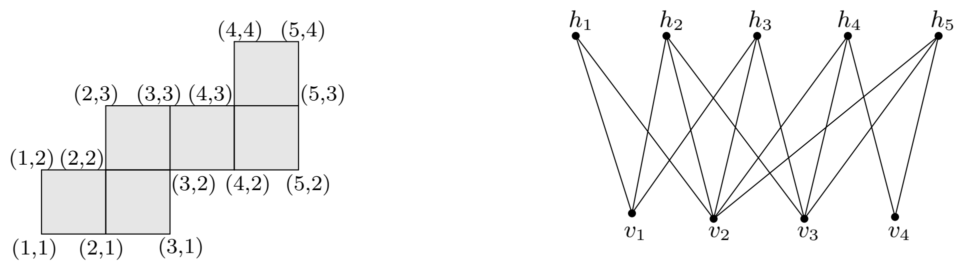

Figure 2.

We let $K[G(\mathcal{P})]$ be the subring of the polynomial ring $T = K[A_{\mathcal{P}} \cup B_{\mathcal{P}}]$ generated by the monomials $h_i v_j$ with $\{h_i, v_j\} \in E(G(\mathcal{P}))$. In other words, $K[G(\mathcal{P})]$ is the edge ring of the bipartite graph $G(\mathcal{P})$. Let, as above, $S = K[x_{ij} : (i,j) \in V(\mathcal{P})]$. As shown in [9], $I_{\mathcal{P}}$ is the kernel of the K-algebra homomorphism $S \to K[G(\mathcal{P})]$ with $x_{ij} \mapsto h_i v_j$. Thus $K[\mathcal{P}] \cong K[G(\mathcal{P})]$, and $K[G(\mathcal{P})]$ is the desired toric parametrization. It is known from [11] that $I_{\mathcal{P}}$ is generated by the binomials corresponding to the cycles in $G(\mathcal{P})$.

By using Corollary 3.5 it is enough to show that for any cycle C of $G(\mathcal{P})$ which has a unique chord, say $e = \{h_i, v_j\}$, there is a crossing path chord of C with respect to e. Since $G(\mathcal{P})$ is a bipartite graph, C is an even cycle, and also $|C| > 4$ because C has a chord. Since every induced cycle of $G(\mathcal{P})$ is a 4-cycle and since C has only one chord, C must be a 6-cycle. Assume that the vertices of C are $h_i, v_{k_1}, h_{\ell_1}, v_j, h_{\ell_2}, v_{k_2}$, listed counterclockwise, and the chord of C is $e = (h_i, v_j)$ as above. With the notation introduced, it follows that

$$(i,j),(i,k_2),(\ell_2,k_2),(\ell_2,j),(\ell_1,j),(\ell_1,k_1,),(i,k_1)$$

are vertices of $\mathcal{P}$. We consider the following cases.

Suppose first that $(\ell_1 - i)(\ell_2 - i) > 0$. Without loss of generality, we may assume $\ell_2 > \ell_1 > i$. Then, since $\mathcal{P}$ is convex and (i,k_2) and (ℓ_2,k_2) are both vertices of $\mathcal{P}$, we have (ℓ_1,k_2) is a vertex of $\mathcal{P}$. It follows that $\{h_{\ell_1}, v_{k_2}\}$ is an edge of $G(\mathcal{P})$ which

is a chord of C, contradicting our assumption that C has a unique chord. Similarly the case that $(k_1 - j)(k_2 - j) > 0$ is also not possible.

It remains to consider the case when $(\ell_1 - i)(\ell_2 - i) < 0$ and $(k_1 - j)(k_2 - j) < 0$. Without loss of generality we may assume that $\ell_1 < i < \ell_2$ and $k_1 < j < k_2$. Then either $(i-1, j+1)$ or $(i+1, j-1)$ is a vertex of $\mathcal{P}$ by the connectedness and convexity of $\mathcal{P}$.

We may assume that $(i-1, j+1) \in V(\mathcal{P})$. Note that $(i-1, k_1)$ and $(\ell_2, j+1)$ belong to $V(\mathcal{P})$. Thus we obtain the path $v_{k_1}, h_{i-1}, v_{j+1}, h_{\ell_2}$ in $G(\mathcal{P})$ which is a crossing path chord of C with respect to e. $\square$

Semi-rigidity. We say that R is *semi-rigid* if $T^1(R)_a = 0$ for all $a \in \mathbb{Z}H$ with $-a \in H$. In this subsection we consider this weak form of rigidity which however is stronger than inseparability.

We again let G be a finite bipartite graph on the vertex set $[m]$ with edge set $E(G) = \{e_1, e_2, \ldots, e_n\}$. The edge ring of G is the toric ring $K[H]$ whose generators are the elements $h_i = \sum_{j \in V(e_i)} \delta_j$, $i = 1, \ldots, n$. Here $\delta_1, \ldots, \delta_m$ is the canonical basis of $\mathbb{Z}^m$. As above we may assume that each edge of G belongs to a cycle and that $C_1, C_2, \ldots C_s$ is the set of cycles of G and where $C_1, \ldots, C_{s_1}$ is the set of induced cycles of G.

Let C_i be one of these cycles with edges $e_{i_1}, e_{i_2}, \ldots, e_{i_{2t}}$ labeled counterclockwise. Two distinct edges e and e' of C_i are said to be of the *same parity in C_i* if $e = e_{i_j}$ and $e' = e_{i_k}$ with $j - k$ an even number.

Lemma 3.7. *Let $a = -h_j - h_k$, and let $i \in [s_1]$. Then $i \in \mathcal{F}'_a$, if and only if e_j and e_k have the same parity in C_i. Moreover, if $\mathcal{F}'_a \neq \emptyset$, then $KL_a = KL_{-h_j} + KL_{-h_k}$.*

Proof. Since $i \in [s_1]$, the cycle C_i is an induced cycle. Let $e_{i_1}, e_{i_2}, \ldots, e_{i_{2t}}$ be the edges of C_i labeled counterclockwise. Then $h(v_i) = \sum_{k=1}^{t} h_{i_{2k-1}} = \sum_{k=1}^{t} h_{i_{2k}}$. Thus if e_j and e_k have the same parity in C_i, it follows that h_j and h_k belong to either one of the above summands, so that $a + h(v_i) \in H$. This shows that $i \in \mathcal{F}'_a$. Conversely, suppose that $i \in \mathcal{F}'_a$. Let $h_j = \delta_{j_1} + \delta_{j_2}$ and $h_k = \delta_{k_1} + \delta_{k_2}$. For simplicity, we may assume that $\delta_1, \ldots, \delta_{2t}$ correspond to the vertices of C_i and that the edges of C_i correspond to the elements $\delta_{2t} + \delta_1$ and $\delta_i + \delta_{i+1}$ for $i = 1, \ldots, 2t-1$. Then $h(v_i) = \delta_1 + \cdots + \delta_{2t}$ and

$$a + h(v_i) = -\delta_{j_1} - \delta_{j_2} - \delta_{k_1} - \delta_{k_2} + \delta_1 + \ldots + \delta_{2t} \in H. \tag{14}$$

In general, let $h \in H$, $h = \sum_{i=1}^{m} z_i \delta_i$ with $z_i \in \mathbb{Z}$. Then it follows that $z_i \geq 0$ for all i. Hence it follows from (14) that e_j and e_k are edges of C_i with $V(e_j) \cap V(e_k) = \emptyset$ (that is, the vertices j_1, j_2, k_1, k_2 are pairwise different), and that $a + h(v_i)$ is the sum of all δ_i, $i = 1, \ldots, 2t$ with $i \neq j_1, j_2, k_1, k_2$. Suppose the edges e_j and e_k do not have the same parity in C_i. Then $a + h(v_i)$ is the sum of S_1 and S_2, where each of S_1 and S_2 consists of an odd sum of δ_i. Hence none of these summands belongs to H. Since $S_1 + S_2 \in H$, there exists a summand δ_{r_1} in S_1 and a summand δ_{r_2} in S_2 such that $\delta_{r_1} + \delta_{r_2} \in H$. This implies that $\{r_1, r_2\} \in E(C_i)$ because C_i has no chord. However this is not possible. Indeed, if $\{r_1, r_2\} \in E(C_i)$, then $r_2 \equiv r_1 + 1 \bmod 2t$. But this is not the case.

Next we show that $KL_a = KL_{-h_j} + KL_{-h_k}$ if $\mathcal{F}'_a \neq \emptyset$. Note that $\mathcal{F}'_a \subseteq \mathcal{F}'_{-h_j} \cap \mathcal{F}'_{-h_k}$, we have $KL_{-h_j} + KL_{-h_k} \subseteq KL_a$ by Lemma 3.1. In order to obtain the desired equality, we only need to show that $v_i \in KL_{-h_j} + KL_{-h_k}$ for each $i \in (\mathcal{F}'_{-h_j} \cap \mathcal{F}'_{-h_k}) \setminus \mathcal{F}'_a$.

Let $i \in (\mathcal{F}'_{-h_j} \cap \mathcal{F}'_{-h_k}) \setminus \mathcal{F}'_a$. Since $\mathcal{F}'_a \neq \emptyset$, there exists an induced cycle, say C, such that e_j and e_k have the same parity in C. We may assume that $V(C) = [2t]$ and $E(C) = \{\{1,2\},\{2,3\},\ldots,\{2t-1,2t\},\{2t,1\}\}$, and that $e_j = \{1,2\}$ and $e_k = \{2k-1,2k\}$ with $1 < k \leq t$. Since e_j, e_k do not have the same parity in C_i, we can assume without loss of generality that $E(C_i)$ is

$$\begin{aligned}&\{\{1,2\},\{2,i_1\},\{i_1,i_2\},\ldots,\{i_{2h},i_{2h+1}\},\{i_{2h+1},2k\},\{2k,2k-1\}\}\\ &\cup\{\{2k-1,i_{2h+2}\},\ldots,\{i_{2\ell},i_{2\ell+1}\},\{i_{2\ell+1},1\}\}.\end{aligned}$$

Then we have even closed walks

$$W_1\colon 2,3,\ldots,(2k-1),2k,i_{2h+1},i_{2h}\ldots,i_1,2$$

and

$$W_2 : 1,2,3,\ldots,(2k-1),i_{2h+2},\ldots,i_{2\ell+1},1.$$

Let $w_1 = v(W_1)$ and $w_2 = v(W_2)$. Since the vertex 1 belongs to e_j but is not a vertex of W_1, Lemma 3.2 implies that $w_1 \in KL_{-h_j}$. Similarly it follows that $w_2 \in KL_{-h_k}$. Since v_i differs at most by a sign from either $w_1 - w_2$ or $w_1 + w_2$, it follows that $v_i \in KL_{-h_j} + KL_{-h_k}$, as required. □

Lemma 3.8. *Suppose that $\mathcal{F}'_{-h_j} \neq \mathcal{F}'_{-h_k}$. Then $KL_{-h_j} \neq KL_{-h_k}$.*

Proof. Let $i \in \mathcal{F}'_{-h_j} \setminus \mathcal{F}'_{-h_k}$. Then $v_i \in KL_{-h_k}$ and $v_i(j) \neq 0$, since e_j is an edge of C_i. However the vectors v which belong to KL_{-h_j} have the property that $v(j) = 0$. Hence $v_i \in KL_{-h_k} \setminus KL_{-h_j}$, and this implies $KL_{-h_j} \neq KL_{-h_k}$. □

Corollary 3.9. *Assume that $K[G]$ is inseparable. Let $a = -h_j - h_k$. Then*

$$\dim_K KL_a = \dim_K KL - 1 \text{ if } \mathcal{F}'_{-a} \neq \emptyset \text{ and } \mathcal{F}'_{-h_j} = \mathcal{F}'_{-h_k}.$$

Otherwise, $\dim_K KL_a = \dim_K KL$.

Proof. Since we assume that G is inseparable, it follows from Corollary 1.1 and Proposition 1.2 that $\dim_K KL - \dim_K KL_{-h_j} = \dim_K(\operatorname{Im}\delta^*)_{-h_j}$. Since by assumption each edge of G belongs to a cycle, it follows that $\dim_K(\operatorname{Im}\delta^*)_{-h_j} = 1$. Thus $\dim_K KL_{-h_j} = \dim_K KL - 1$. Similarly, $\dim_K KL_{-h_k} = \dim_K KL - 1$. If $\mathcal{F}'_{-h_j} = \mathcal{F}'_{-h_k}$, then $KL_{-h_j} = KL_{-h_k}$, and if moreover, $\mathcal{F}'_{-a} \neq \emptyset$, then together with Lemma 3.7 we have $\dim_K KL_a = \dim_K KL - 1$, as desired.

Otherwise, there are two cases to consider. If $\mathcal{F}'_{-a} = \emptyset$, then $KL_a = KL$, by the definition of KL_a and by Lemma 3.1. If $\mathcal{F}'_{-a} \neq \emptyset$ and $\mathcal{F}'_{-h_j} \neq \mathcal{F}'_{-h_k}$, then $KL_a = KL_{-h_j} + KL_{-h_k} = KL$, using Lemma 3.7 together with Lemma 3.8. □

Theorem 3.10. *Let G be a bipartite graph such that $R = K[G]$ is inseparable. Then the following statements are equivalent:*

(a) *$K[G]$ is not semi-rigid;*

(b) *there exist edges e, f and an induced cycle C such that e, f have the same parity in C and for any other induced cycle C', $e \in E(C')$ if and only if $f \in E(C')$.*

Proof. (b) $\Rightarrow$ (a): Let $a = -g - h$, where g and h are vectors in H corresponding to the edges e and f respectively. Then $\dim_K KL_a = \dim_K KL - 1$ by Corollary 3.9. Note that $\mathcal{G}_a = \emptyset$, we have $(\operatorname{Im} \delta^*)_a = 0$. Therefore $T^1(R)_a \neq 0$ by Corollary 1.3, and in particular, R is not semirigid.

(a) $\Rightarrow$ (b): By assumption, there exists $a = \sum_{i \in [n]} -a_i h_i \in \mathbb{Z}H$ with $a_i \geq 0$ for $i = 1, \cdots, n$ such that $T^1(R)_a \neq 0$. Note that $a_i \in \{0, 1\}$, for otherwise, $\mathcal{F}'_a = \emptyset$ and so $KL_a = KL$. In particular $T^1(R)_a = 0$, a contradiction. Since R is inseparable, it follows that $|\{i\colon\ a_i \neq 0\}| \geq 2$. If $|\{i\colon\ a_i \neq 0\}| = 2$, then $a = -h_k - h_j$ for some $1 \leq i \neq j \leq n$. Therefore, $\mathcal{F}'_a \neq \emptyset$ and $\mathcal{F}'_{-h_j} = \mathcal{F}'_{-h_k}$ by Corollary 1.1 and Corollary 3.9.

Let e and f be the edges corresponding to the vectors h_j and h_k, respectively. Then, since $\mathcal{F}'_a \neq \emptyset$, there exists an induced cycle C of G such that e and f have the same parity in C, by Lemma 3.7. Moreover, $\mathcal{F}'_{-h_j} = \mathcal{F}'_{-h_k}$ implies that for any induced cycle C' of G, $e \in E(C')$ if and only if $f \in E(C')$.

Now suppose that $|\{i\colon\ a_i \neq 0\}| \geq 3$. Then there exists j and k with $a_j \neq 0$ and $a_k \neq 0$, and we set $b = -h_j - h_k$. Note that $\mathcal{F}'_a \subseteq \mathcal{F}'_b$. This implies that $KL_b \subseteq KL_a$. Therefore, since $(\operatorname{Im} \delta^*)_a = (\operatorname{Im} \delta^*)_b = 0$, we have $T^1(R)_b \neq 0$, and we are in the previous case. □

Corollary 3.11. *Let $\mathcal{P}$ be a convex polyomino. Then $K[\mathcal{P}]$ is semi-rigid if and only if $\mathcal{P}$ contains more than one cell.*

Proof. Assume that $\mathcal{P}$ contains a unique cell. Then $G(\mathcal{P})$ is a square and it is not semi-rigid by Theorem 3.10.

Conversely, assume that $K[\mathcal{P}]$ is not semi-rigid. Then there exist two edges e, f and an induced cycle C of $G(\mathcal{P})$ satisfying the condition (b) in Theorem 3.10. Let (i, j) and (k, ℓ) be vertices of $\mathcal{P}$ corresponding to the edge e and f, respectively. Then the two edges of C other than e and f correspond to the vertices (i, ℓ) and (k, j) of $\mathcal{P}$. It follows that $k \neq i$ and $\ell \neq j$. Without loss of generality, we may assume that $k > i$ and $\ell > j$. Then $(i+1, j+1) \in V(\mathcal{P})$. Let C' be the induced cycle of $G(\mathcal{P})$ corresponding to the cell $[(i, j), (i+1, j+1)]$ of $\mathcal{P}$. Since C' contains the edge e, C' must contain f by the condition (b) and thus $k = i+1$ and $\ell = j+1$. We claim that $[(i, j), (i+1, j+1)]$ is the only cell of $\mathcal{P}$. Suppose that this is not the case. Then we let C_t, $t = 1, 2, 3, 4$ be four cells which share a common edge with the cell $[(i, j), (i+1, j+1)]$. Note that $\mathcal{P}$ contains at least one of the C_t. Indeed, since $\mathcal{P}$ is connected and since by assumption $\mathcal{P}$ contains a cell C different from $[(i, j), (i+1, j+1)]$, there exists a path in $\mathcal{P}$ between the cell $[(i, j), (i+1, j+1)]$ and C. This path must contain one of the C_t. However $V(C_t)$ contains exactly one of the two vertices (i, j) and $(i+1, j+1)$ for $t = 1, \ldots, 4$. In other words, there exists an induced cycle of $G(\mathcal{P})$ which contains exactly one of the edges e and f. This is contradicted to the condition (b) and thus our claim has been proved. □

Classes of bipartite graphs which are semi-rigid or rigid. Let G_n be the bipartite graph on vertex set $[2n]$ with edge set

$$E(G_n) = \{\{i, j\}\colon\ i, j \in [2n], i - j \text{ is odd and } \{i, j\} \neq \{1, 2n\}\}.$$

Thus G_n is obtained from the complete bipartite graph $K_{n,n}$ by deleting one of its edges. We observe that for $n \geq 3$, $G_n = G(\mathcal{P}_n)$ where $\mathcal{P}_n$ is the polyomino with

$$V(\mathcal{P}_n) = \{(i, j)\colon\ 1 \leq i, j \leq n, (i, j) \neq (n, n)\}.$$

For any edge $e = \{i, j\} \in E(G)$ we use $h(e)$ to denote the vector $\delta_i + \delta_j \in \mathbb{Z}^{2n}$. Let C be a cycle. Then, as before, $v(C)$ stands for the vector corresponding to C and $V(C)$ denotes the set of vertices of C. Our main result of this subsection is the following:

Proposition 3.12. *Let R be the edge ring of G_n.*
 (a) *If $n = 3$, then R is semirigid, but not rigid.*
 (b) *If $n \geq 4$, then R is rigid.*

We need some preparations. First, we introduce some notation.

Let $a = \sum_{i\in[2n]} a_i\delta_i \in \mathbb{Z}^{2n}$ with $a_i \in Z$ for $i = 1, \ldots, 2n$. We set

$$a_e = \sum_{i \text{ is even}} a_i \qquad \text{and} \qquad a_o = \sum_{i \text{ is odd}} a_i.$$

We also set

$$\ell(a) = a_1 + a_{2n} \qquad \text{and} \qquad r(a) = \sum_{i\notin\{1,2n\}} a_i.$$

Lemma 3.13. *Let $a = \sum_{i\in[2n]} a_i\delta_i$ and $H = H(G_n)$. Then*
 (a) *$a \in \mathbb{Z}H$ if and only if $a_e = a_o$.*
 (b) *The following conditions are equivalent:*
  (i) *$a \in H$;*
  (ii) *$a_e = a_o$, $\ell(a) \leq r(a)$ and $a_i \geq 0$ for all $i = 1, \ldots, 2n$.*

Proof. (a) First note that if i is even and j is odd, then $\delta_i + \delta_j \in \mathbb{Z}H$. Indeed, if $\{i, j\} \neq \{1, 2n\}$, then $\delta_i + \delta_j \in H$; if $\{i, j\} = \{1, 2n\}$, then $\delta_1 + \delta_{2n} = (\delta_1 + \delta_2) + (\delta_3 + \delta_{2n}) - (\delta_2 + \delta_3) \in \mathbb{Z}H$.

Suppose that $a_e = a_o$. We prove that $a \in \mathbb{Z}H$ by induction on $|a_e|$. If $a_e = 0$ then $a = 0$, and the assertion is trivial. Suppose that $|a_e| > 0$. We only consider the case when $a_e > 0$ since the other case is similar. In this case there exist $i, j \in [2n]$ such that i is even, j is odd, $a_i > 0$ and $a_j > 0$. Let $b = a - (\delta_i + \delta_j)$. Then $b_e = a_e - 1 = a_o - 1 = b_o$, and so $b \in \mathbb{Z}H$ by induction. This implies that $a = b + (\delta_i + \delta_j) \in \mathbb{Z}H$. Conversely, it is obvious that $a_e = a_o$ if $a \in \mathbb{Z}H$.

(b) (i) $\Rightarrow$ (ii): Note that $\ell(h(e)) \leq r(h(e))$ for any $e \in E(G)$ since $\{1, 2n\} \notin E(G)$. Now given $a \in H$. Then $a = \sum_{e\in E(G)} c_e h(e)$, where c_e is a non-negative integer for each $e \in E(G)$. It follows that $\ell(a) = \sum_{e\in E(G)} c_e\ell(h(e)) \leq \sum_{e\in E(G)} c_e r(h(e)) = r(a)$, as required.

(ii) $\Rightarrow$ (i): We use induction on $\ell(a)$. If $\ell(a)=0$, we see that $a\in H$ by induction on a_e as in the proof of (a). Assume that $\ell(a)>0$. Without restriction we may further assume that $a_1\geq a_{2n}$. Then

$$a_e-a_{2n}=\sum_{\substack{i\in even,\\ i\neq 2n}} a_i>0,$$

for otherwise $a_e=a_{2n}\leq a_1<a_o$, a contradiction. (For the inequality $a_1<a_o$ we used that $0<\ell(a)\leq r(a)=a_3+a_5+\cdots+a_{2n-1}$.) Hence there exists an even number $j\neq 2n$ with $a_j>0$. Set $b=a-(\delta_1+\delta_j)$. Then $b\in H$ by induction and so $a=b+(\delta_1+\delta_j)\in H$. $\square$

Corollary 3.14. *Let $a\in\mathbb{Z}H$ with $a_i\geq 0$ for all $i\in[2n]$. Then either $a\in H$ or $a=b+k(\delta_1+\delta_{2n})$, where $k\geq 1$ and $b\in H$ with $\ell(b)=r(b)$.*

Proof. Suppose that $a\notin H$. Then $\ell(a)>r(a)$, by Lemma 3.13. Note that by Lemma 3.13 we have $a_e=a_o$, and hence $\ell(a)-r(a)=a_1+a_{2n}-(a_e-a_{2n})-(a_o-a_1)$ is an even number, say $2k$. Set $b=(a_1-k)\delta_1+(a_{2n}-k)\delta_{2n}+\sum_{i\notin\{1,2n\}}a_i\delta_i$. Suppose that $a_1<k$. Then

$$a_{2n}=2k-a_1+\sum_{i\notin\{1,2n\}}a_i>a_1+\sum_{i\notin\{1,2n\}}a_i\geq a_o,$$

a contradiction, because $a_e=a_o$. Therefore $a_1\geq k$. Similarly, $a_{2n}\geq k$. Since $\ell(b)=r(b)$, Lemma 3.13 implies that $b\in H$. Moreover, $a=b+k(\delta_1+\delta_{2n})$ by the choices of k and b. $\square$

Lemma 3.15. *Let $a\in\mathbb{Z}H$ such that $E(G_n)\setminus\{e\colon\ a+h(e)\in H\}$ contains no cycle. Then $\dim_K KL=\dim_K D_a$. In particular, $T^1(R)_a=0$.*

Proof. Let Γ' be the graph with $E(\Gamma')=E(G_n)\setminus\{e\colon\ a+h(e)\in H\}$, and let Γ be the graph obtained from Γ' by adding all vertices of G_n which do not belong to $V(\Gamma')$. Then Γ is a graph with no cycle and $V(\Gamma)=[2n]$. If Γ is a tree, then Γ is a spanning tree of G_n. If Γ is not a tree, we choose a connected component Γ_0 of Γ and let Γ_1 be the induced graph of Γ on the set $[2n]\setminus V(\Gamma_0)$. Since G_n is connected there exists an edge e of G_n with one end in $V(\Gamma_0)$ and the other end in $V(\Gamma_1)$. We let Γ'' be the graph obtained from Γ by adding the edge e. Since neither Γ_0 nor Γ_1 contains a cycle, it follows that Γ'' has also no cycle, but one more edge than Γ. Proceeding in this way we obtain after finitely many steps a spanning tree T of G_n which contains $E(G_n)\setminus\{e\colon\ a+h(e)\in H\}$. In particular, there exists a subset of $\{e\colon\ a+h(e)\in H\}$, say $\{e_1,\dots,e_k\}$ such that $E(T)=E(G_n)\setminus\{e_1,e_2,\dots,e_k\}$. Since all spanning trees have the same number of edges, namely, $2n-1$, it follows that $k=n^2-2n$.

For each $i=1,\dots,k$, $T+e_i$ contains a unique induced cycle, say C_i. Then for all $i=1,\dots,k$ we have $v_i(i)\in\{\pm1\}$ and $v_i(j)=0$ if $j\neq i$ and $1\leq j\leq k$. Here $v_i=v(C_i)$, the vector corresponding to the cycle C_i, for $i=1,\dots,k$. It follows that $\dim_K D_a\geq k$ since $(v_1(i),\dots,v_k(i),\dots,v_{s_1}(i))\in D_a$ for $i=1,\dots,k$. Here s_1 is the number of induced cycles of G_n. On the other hand, $\dim_K D_a\leq\dim_K KL$ and

$\dim_k KL = |E(G_n)| - |V(G_n)| + 1$ which is equal to $k = n^2 - 2n$. It follows that $\dim_K KL = \dim_K D_a$, completing the proof by Proposition 1.2. $\square$

Proof of Proposition 3.12. (a) Since G_3 is associated with a polyomino containing 8 vertices, we have $R(= K[G_3])$ is semi-rigid by Corollary 3.11.

Let $a = \delta_6 - \delta_1 - \delta_2 - \delta_4$. Then KL_a is spanned by the vectors corresponding to the cycles $C_1 : 3, 2, 5, 4$, $C_2 : 3, 4, 5, 6$ and $C_3 : 3, 2, 5, 6$. This implies that $\dim_K KL_a = 2$. Since $\dim_K D_a = 0$ and $\dim_K KL = 3$, we have $T^1(R)_a \neq 0$. In particular, R is not rigid, as required.

(b) Let $a = \sum_{i\in[2n]} a_i\delta_i \in \mathbb{Z}H$. We want to prove that $T^1(R)_a = 0$. There are following cases to consider.

Case 1 : $a_i \geq 0$ for all i. By Corollary 3.14, either $a \in H$ or $a = b + k(\delta_1 + \delta_{2n})$, where $k \geq 1$ and $b \in H$ with $\ell(b) = r(b)$. If $a \in H$, then $T^1(R)_a = 0$, see Corollary 1.4. If $a = b + k(\delta_1 + \delta_{2n})$ with $k = 1$, then for any edge $e = \{i, j\}$ with $e \cap \{1, 2n\} = \emptyset$, we have $a + \delta_i + \delta_j \in H$ by Lemma 3.13. It follows that $E(G) \setminus \{e\colon\ a + h(e) \in H\}$ contains no cycle, and so $T^1(R)_a = 0$ by Lemma 3.15. If $k = 2$, then for any induced cycle C, $a + h(v(C)) \in H$ if and only if $V(C) \cap \{1, 2n\} = \emptyset$. This follows from Lemma 3.13 and the fact that any induced cycle of G_n is a 4-cycle. To prove $KL = KL_a$, we have to show if $V(C) \cap \{1, 2n\} = \emptyset$, then $v(C) \in KL_a$. Given an induced cycle $C : i_1, i_2, i_3, i_4$ with $V(C) \cap \{1, 2n\} = \emptyset$, where i_1, i_3 are even and i_2, i_4 are odd. Then we obtain two cycles $C_1\colon\ i_1, i_2, i_3, 1$ and $C_2 : i_3, i_4, i_1, 1$. Note that $v(C_1), v(C_2) \in KL_a$ and $v(C)$ is a linear combination of $v(C_1), v(C_2)$, we have $KL = KL_a$ and so $T^1(R)_a = 0$. If $k \geq 3$, then for any induced cycle C, one has $a + h(v(C)) \notin H$ by Lemma 3.13 and so $KL_a = KL$. In particular, $T^1(R)_a = 0$.

Case 2: There exists a unique $i \in [2n]$ with $a_i < 0$. If $a_i \leq -2$, then $\mathcal{F}'_a = \emptyset$ and so $KL_a = KL$. In particular $T^1(R)_a = 0$. Hence we assume $a_i = -1$. By symmetry, we only need to consider the cases when $i = 1$ and when $i = 3$.

We first assume that $i = 1$. Since $a_e = a_o$, there exists an odd integer $j \neq 1$ such that $a_j > 0$, and so $a = b + \delta_j - \delta_1$, where $b_e = b_o$ and $b_\ell \geq 0$ for each $\ell \in [2n]$. By Corollary 3.14, either $b \in H$ or $b = c + k(\delta_1 + \delta_{2n})$ with $c \in H$ and $k > 0$. The second case cannot happen because $a_1 = -1$. Hence for any $e \in E(G)$, $a + h(e) \in H$ if and only if $1 \in e$. In other words, $a + h(e) \in H$ if and only if $e \in \{\{1, 2\}, \{1, 4\}, \ldots, \{1, 2n - 2\}\}$. Denote $\{1, 2i\}$ by e_i for $i = 1, \ldots, n - 1$. Let C_i be the cycle $1, 2i, 3, 2n - 2$ and let $v_i = v(C_i)$ for $i = 1, \ldots, n - 2$. Then for $i = 1, \ldots, n - 2$, we have $v_i(i) \in \{\pm 1\}$ and $v_i(j) = 0$ for $j \neq i$ and $j = 1, \ldots, n - 2$. This implies that $\dim_K D_a \geq n - 2$. To compute $\dim_K KL_a$, we notice that if C is an induced cycle with $1 \notin V(C)$, then $a + h(v(C)) \notin H$ and thus KL_a contains the cycle space of the complete bipartite graph with bipartition $\{3, 5, \ldots, 2n - 1\}$ and $\{2, \ldots, 2n\}$, which has the dimension $(n - 1)n - n - (n - 1) + 1 = n^2 - 3n + 2$, see (10). Thus $T^1(R)_a = 0$ because $\dim_K D_a = n^2 - 2n$.

Next we assume that $i = 3$. Then $a = b + \delta_j - \delta_3$, where $b_\ell \geq 0$ for all $\ell \in [2n]$ and $b_e = b_o$, $j \neq 3$ and j is odd. Moreover by Corollary 3.14, we have either $b \in H$ or $b = c + k(\delta_1 + \delta_{2n})$ for some $k \geq 1$ and with $c \in H$ and $\ell(c) = r(c)$. Suppose first that $b \notin H$ and $k \geq 2$. Then for any cycle C, $a + h(v(C)) \in H$

implies $V(C) \cap \{1, 2n\} = \emptyset$. Thus, similarly as in Case 1 we see that $KL_a = KL$ and $T^1(R)_a = 0$. Suppose next that $j \neq 1$ and that $b \in H$ or $b \notin H$ and $k = 1$. Then $a + h(e) \in H$ for any $e \in \{\{3, 2\}, \dots, \{3, 2n-2\}\}$. Denote $\{3, 2t\}$ by e_t for $t = 1, \dots, n-1$. For $t = 1, \dots, n-1$, let C_t be the cycle $3, 2t, 5, 2n$ and let $v_t = v(C_t)$, the vector corresponding to C_t. Then $v_t(t) \in \{\pm 1\}$ for $t = 1, \dots, n-1$ and $v_t(k) = 0$ for $k \neq t$. This implies that $\dim_K D_a = n - 1$. On the other hand, KL_a contains the cycle space of the subgraph of G_n induced on $\{1, 5, \dots, 2n-1\} \cup \{2, 4, \dots, 2n\}$, which has the dimension $n^2 - 3n + 1$. Thus $T^1(R)_a = 0$.

Finally suppose that $j = 1$ and that and also $k = 1$ if $b \notin H$. If $b \in H$, then we check that $a + h(e) \in H$ for any $e \in \{\{3, 2\}, \dots, \{3, 2n-2\}\}$ and deduce that $T^1(R)_a = 0$, in the same process as in the last case. If $b \notin H$ and $k = 1$, then for any induced cycle C, we have $a + h(v(C)) \in H$ if and only if $3 \in V(C)$ and $\{1, 2n\} \cap V(C) = \emptyset$. We claim that $KL_a = KL$. Given an induced cycle $C : 3, i_1, i_2, i_3$ with $a + h(v(C)) \in H$. Here i_1 and i_3 are even and i_2 is odd. We let $C_1 : 3, i_1, i_2, 2n$ and $C_2 : i_2, i_3, 3, 2n$. Then $v(C_1)$ and $v(C_2)$ belong to KL_a and $v(C)$ is a linear combination of $v(C_1)$ and $v(C_2)$. Thus $KL_a = KL$, as claimed. In particular, $T^1(R)_a = 0$.

Case 3: $|\{k\colon\ a_k < 0\}| = 2$. Without restriction we may assume $a_i = a_j = -1$ for some $i \neq j$. Indeed, if some $a_k \leq -2$, then $\mathcal{F}'_a = \emptyset$ by Lemma 3.13 and so $T^1(R)_a = 0$. Assume first that both i and j are even. Then for any induced cycle C such that $\{i, j\} \nsubseteq V(C)$, we have $v(C) \in KL_a$. Let $C : k, i, \ell, j$ be a cycle with $\{i, j\} \subseteq V(C)$. We choose an even number $d \in [2n] \setminus \{i, j, 2n\}$. Then we obtain two cycles $C_1 : k, i, \ell, d$ and $C_2 : \ell, j, k, d$. Since $v(C)$ is a linear combination of $v(C_1)$ and $v(C_2)$ and since $v(C_t) \in KL_a$ for $t = 1, 2$, we have $v(C) \in KL_a$ and thus $KL_a = KL$. In particular $T^1(R)_a = 0$.

Next assume that i is even and j is odd and $\{i, j\} \neq \{1, 2n\}$. Notice that we can write a as $a = b + k(\delta_1 + \delta_{2n}) - (\delta_i + \delta_j)$, where $b \in H$ and $k \geq 0$. Moreover, if $k > 0$ then $\ell(b) = r(b)$.

If $k = 0$, then $\dim_K D_a = 1$, and KL_a contains the cycle space of the graph which is obtained from G_n by deleting the edge $\{i, j\}$. Hence $\dim_K KL_a \geq \dim_K KL - 1$, and so $T^1(R)_a = 0$.

If $k = 1$, then for any induced cycle C, we have $a + h(v(C)) \in H$ if and only if $\{i, j\} \subseteq V(C)$ and $V(C) \cap \{1, 2n\} = \emptyset$. Let $C : i, j, k, \ell$ be an induced cycle such that $a + h(v(C)) \in H$. Then the vectors v_1, v_2 which correspond to cycles $j, k, \ell, 2n$ and $\ell, i, j, 2n$ respectively belong to KL_a and $v(C)$ is a linear combination of v_1, v_2. It follows that $KL = KL_a$ and $T^1(R)_a = 0$.

If $k \geq 2$, we first note that $\{i, j\} \cap \{1, 2n\} = \emptyset$. Indeed, if $\{i, j\} \cap \{1, 2n\} \neq \emptyset$, then either $a_i \geq 1$ or $a_j \geq 1$, a contradiction. Thus $\mathcal{F}'_a = \emptyset$ by Lemma 3.13 and it follows that $KL = KL_a$. In particular $T^1(R)_a = 0$.

Finally assume that $\{i, j\} = \{1, 2n\}$. Then $a = b - \delta_1 - \delta_{2n}$ with $b \in H$ and so $\mathcal{F}'_a = \emptyset$. It follows that $KL_a = KL$ and $T^1(R)_a = 0$.

Case 4: $|\{k\colon\ a_k < 0\}| = 3$. We may assume that $a_i = a_j = a_k = -1$. We only need to consider the case when $\mathcal{F}'_a \neq \emptyset$. So we may assume i, k are even and j is odd, and $\{1, 2n\} \nsubseteq \{i, j, k\}$. Let $C : i, j, k, \ell$ be an induced cycle such that

$a+h(v(C)) \in H$. We choose an even number $d \in [2n] \setminus \{i,k,2n\}$ and let $C_1 : j,k,\ell,d$ and $C_2 : \ell,i,j,d$ be two cycles in G_n. Then $v(C_1)$ and $v(C_2)$ belong to KL_a and $v(C)$ is a linear combination of $v(C_1)$ and $v(C_2)$. This implies $KL_a = KL$, and in particular, $T^1(R)_a = 0$.

Case 5: $|\{k\colon\ a_k < 0\}| \geq 4$. If $|\{k\colon\ a_k < 0\}| = 4$, we may assume that $a_i = a_j = a_k = a_\ell = -1$. Then for any induced cycle C, $a + h(v(C)) \in H$ implies that $V(C) = \{i,j,k,\ell\}$. We may assume that i and k are even numbers. Let t be an odd number in $[2n] \setminus \{j,\ell\}$, and let $C_1 : i,j,k,t$ and $C_2 : k,l,i,t$ be 4-cycles of G_n. Since $v(C)$ is a linear combination of $v(C_1)$ and $v(C_2)$, we have $KL_a = KL$, and consequently, $T^1(R)_a = 0$. If $|\{k\colon\ a_k < 0\}| > 4$, then $\mathcal{F}'_a = \emptyset$ and so $T^1(R)_a = 0$.

Thus we have shown that $T^1(R)_a =$ for all a, and this shows that R is rigid, as desired. $\square$

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